VOLUME 31, NUMBER 1, MARCH 2014

ALGEBRAS, GROUPS AND GEOMETRIES

ALGEBRAS, GROUPS & GEOMETRIES Volume 31, Number 1, March 2014

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ON SANTILLI ISOFUNCTOR FOR MATHEMATICAL ISOSTRUCTURES*

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Received February 24, 2014

Abstract

The most important mathematical iso-structures are geometric, algebraic and topological ones [1–5].

The objects of modern iso–Geometry, iso-spaces, like all objects of modern iso-Mathematics, are sets of elements of arbitrary nature endowed with some mathematical iso-structure, for example, iso-symmetry etc.

The typical way to think about iso-symmetry is with the concept of a "iso-group". But to get a concept of iso-symmetry that's really up to the demands put on it by modern iso-Mathematics, we need — at the very least — to work with a "category" of iso-symmetries, rather than a iso-group of iso-symmetries. In this article we construct Santilli isofunctor for mathematical iso-structures. The compositions of morphisms between Santilli isofunctors are built in terms of diagrams and equations. There are obtained categories of iso-groups, iso-rings, iso-fields, vector iso-spaces, iso-algebras, etc.

*This research has been partially supported by the R. M. Santilli Foundation.

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1 Categories, monoids and groupoids

Category theory groups together in categories the mathematical objects with some common structure (e.g., sets, partially ordered sets, groups, rings, and so forth) and the appropriate morphisms between such objects [6–12].

DEFINITION 1.1. *A category is a quadruple* $(\mathrm{Ob}, \mathrm{Hom}, \mathrm{id}, \circ)$ *consisting of:*

(Cl) a class Ob *of objects;*

(C2) for each ordered pair (A, B) *of objects a set* $\mathrm{Hom}(A, B)$ *of morphisms;*

(C3) for each object A *a morphism* $\mathrm{id}_A \in \mathrm{Hom}(A, A)$*, the identity of* A*;*

(C4) a composition law associating to each pair of morphisms $f \in \mathrm{Hom}(A, B)$ *and* $g \in \mathrm{Hom}(B, C)$ *a morphism* $g \circ f \in \mathrm{Hom}(A, C)$*;*

which is such that:

(M1) $h \circ (g \circ f) = (h \circ g) \circ f$ *for all* $f \in \mathrm{Hom}(A, B)$*,* $g \in \mathrm{Hom}(B, C)$ *and* $h \in \mathrm{Hom}(C, D)$*;*

(M2) $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ *for all* $f \in Hom(A, B)$*;*

(M3) the sets $\mathrm{Hom}(A, B)$ *are pairwise disjoint.*

This last axiom is necessary so that given a morphism we can identify its domain A and codomain B, however it can always be satisfied by replacing $\mathrm{Hom}(A, B)$ by the set $\mathrm{Hom}(A, B) \times (\{A\}, \{B\})$.

EXAMPLE 1.1. *The classic example is* **Set**, *the category with sets as objects and functions as morphisms, and the usual composition of functions as composition.*

To understand this definition, we should ask first: what is a category with one object? It is a — *monoid.* We shall also give the definition of a monoid in §3 in more details like this: a set M with an associative binary product and a unit element 1 such that $a1 = 1a = a$ for all a in M. Monoids abound in mathematics; they are in a sense the most primitive interesting algebraic structures.

To check that a category with one object is "essentially just a monoid", note that if our category $\mathbf{C}$ has one object x, the set $\mathrm{Hom}(x,x)$ of all morphisms from x to x is indeed a set with an associative binary product, namely composition, and a unit element, namely id_x.

How about categories in which every morphism is invertible? We say a morphism $f : x \to y$ in a category has inverse $g : y \to x$ if $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. Well, a category in which every morphism is invertible is called a *groupoid.*

Finally, a group is a category with one object in which every morphism is invertible. It's both a monoid and a groupoid!

When we use groups in physics to describe symmetry, we think of each element g of the group G as a "process". The element 1 corresponds to the "process of doing nothing at all". We can compose processes g and h — do h and then g — and get the product $g \circ h$. Crucially, every process g can be "undone" using its inverse g^{-1}.

So: a monoid is like a group, but the "symmetries" no longer need be invertible; a category is like a monoid, but the "symmetries" no longer need to be composable.

A morphism $a\colon A \to B$ is called *isomorphism* if there exists a morphism $b\colon B \to A$ such that $a \circ b = i_B$ and $b \circ a = i_A$. In this case objects A and B are called *isomorphic.*

Morphisms $a\colon D \to A$ and $b\colon D \to B$ are called *isomorphic* if there exists an isomorphism $c\colon A \to B$ such that $c \circ a = b$.

An object Z is called *terminal* object if for any object A there exists a unique morphism from A to Z, which is denoted $z_A\colon A \to Z$ in what follows.

It is said that a category has (pairwise) products if for every pair of objects A and B there exists their *product*, that is, an object $A \times B$ and a pair of morphisms $\pi_{A,B}\colon A\times B \to A$ and $\nu_{A,B}\colon A\times B \to B$, called projections, such that for any object D and for any pair of morphisms $a\colon D \to A$ and $b\colon D \to B$ there exists a unique morphism $c\colon D \to A \times B$, satisfying the following conditions:

$$\pi_{A,B} \circ c = a, \qquad \nu_{A,B} \circ c = b. \tag{1}$$

We call such morphism c the *product of morphisms* a and b and denote it $a * b$.

It is easily seen that existence of products in a category implies the following equality:

$$(a * b) \circ d = (a \circ d) * (b \circ d). \tag{2}$$

In a category with products, for two arbitrary morphisms $a\colon A \to C$ and $b\colon B \to D$ one can define the morphism $a \times b$:

$$a \times b\colon A \times B \to C \times D, \qquad a \times b \stackrel{\text{def}}{=} (a \circ \pi_{A,B}) * (b \circ \nu_{A,B}). \tag{3}$$

This definition and (1) obviously imply that the morphism $c = a \times b$ satisfy the following conditions:

$$\pi_{C,D} \circ c = a \circ \pi_{A,B}, \qquad \nu_{C,D} \circ c = b \circ \nu_{A,B}. \tag{4}$$

Moreover, $c = a \times b$ is the only morphism satisfying conditions (4).

It is also easily seen that (2) and (3) imply the following equality:

$$(a \times b) \circ (c * d) = (a \circ c) * (b \circ d). \tag{5}$$

Suppose $A \times B$ and $B \times A$ are two products of objects A and B taken in different order. By the properties of products, the objects $A \times B$ and $B \times A$ are isomorphic and the natural isomorphism is

$$\sigma_{A,B}\colon A \times B \to B \times A, \qquad \sigma_{A,B} \stackrel{\text{def}}{=} \nu_{A,B} * \pi_{A,B}. \tag{6}$$

Moreover, for any object D and for any morphisms $a\colon D \to A$ and $b\colon D \to B$, the morphisms $a * b$ and $b * a$ are isomorphic, that is,

$$\sigma_{A,B} \circ (a * b) = b * a. \tag{7}$$

Similarly, by the properties of products, the objects $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic. Let

$$\alpha_{A,B,C}\colon (A \times B) \times C \to A \times (B \times C)$$

be the corresponding natural isomorphism. Its "explicit" form is:

$$\alpha_{A,B,C} \stackrel{\text{def}}{=} (\pi_{A,B} \circ \pi_{A\times B,C}) * \Big((\nu_{A,B} \circ \pi_{A\times B,C}) * \nu_{A\times B,C}\Big). \tag{8}$$

Then for any object D and for any morphisms $a\colon D \to A$, $b\colon D \to B$, and $c\colon D \to C$ we have

$$\alpha_{A,B,C} \circ ((a * b) * c) = a * (b * c). \tag{9}$$

REMARK 1.1. *If* $\mathbf{A} = (\mathrm{Ob}, \mathrm{Hom}, \mathrm{id}, \circ)$ *is a category, then*
1) The class Ob *of* $\mathbf{A}$*-objects is usually denoted by* $\mathrm{Ob}(\mathbf{A})$.
2) The class of all $\mathbf{A}$*-morphisms (denoted by* $\mathrm{Mor}(\mathbf{A})$*) is defined to be the union of all the sets* $\mathrm{Hom}(A, B)$ *in* $\mathbf{A}$.
3) If $A \xrightarrow{f} B$ *is an* $\mathbf{A}$*-morphism, we call* A *the* **domain** *of* f *[and denote it by* $dom(f)$*] and call* B *the* **codomain** *of* f *[and denote it by* $cod(f)$*]. Observe that condition (M3) guarantees that each* $\mathbf{A}$*-morphism has a* **unique** *domain and a* **unique** *codomain. However, this condition is given for technical convenience only, because whenever all other conditions are satisfied, it is easy to "force" condition (M3) by simply replacing each morphism* f *in* $\mathrm{Hom}(A, B)$ *by a triple* (A, f, B). *For this reason, when verifying that an entity is a category, we will disregard condition (M3).*
4)The composition, $\circ$, *is a partial binary operation on the class* $\mathrm{Mor}(\mathbf{A})$. *For a pair* (f, g) *of morphisms,* $f \circ g$ *is defined if and only if the domain of* f *and the codomain of* g *coincide.*
5) If more than one category is involved, subscripts may be used for clarification (as in $\mathrm{Hom}_{\mathbf{A}}(A, B)$*).*

2 Functors and natural transformations

DEFINITION 2.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories. A* **functor** *from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is a family of functions* $\mathcal{F}$ *which associates to each object* A *in* $\mathbf{X}$ *an object* $\mathcal{F}A$ *in* $\mathbf{Y}$ *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\mathcal{F}f \in \mathrm{Hom}_{\mathbf{Y}}(\mathcal{F}A, \mathcal{F}B)$, *and which is such that:*

(FI) $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\mathcal{F}\,\mathrm{id}_A = \mathrm{id}_{\mathcal{F}A}$ *for all* $A \in \mathrm{Ob}\,(\mathbf{X})$.

There is the definition of left and right adjoint functors. In the following we shall need two such adjoint constructions. First, in a given category the left adjoint of the diagonal functor (if it exists) is called the coproduct and the right adjoint (if it exists) is called the product: in **Sets** the product is the Cartesian product and the coproduct is the disjoint union.

To the next higher level of abstraction belongs the notion of a *natural transformation.* It is a kind of a function between functors and is defined as follows.

DEFINITION 2.2. *Let* $\mathcal{F} : \mathbf{X} \to \mathbf{Y}$ *and* $\mathcal{G} : \mathbf{X} \to \mathbf{Y}$ *be two functors. A* **natural transformation** $\alpha : \mathcal{F} \to \mathcal{G}$ *is given by the following data:*

For every object A *in* $\mathbf{X}$ *there is a morphism* $\alpha_A : \mathcal{F}(A) \to \mathcal{G}(A)$ *in* $\mathbf{Y}$ *such that for every morphism* $f : A \to B$ *in* $\mathbf{X}$ *the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\alpha_A} & \mathcal{G}(A) \\ \mathcal{F}(f)\downarrow & & \downarrow\mathcal{G}(f) \\ \mathcal{F}(B) & \xrightarrow{\alpha_B} & \mathcal{G}(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $\mathcal{G}(f) \circ \alpha_A = \alpha_B \circ \mathcal{F}(f)$.

The morphisms α_A, $A \in \mathrm{Ob}(\mathbf{X})$, are called the *components of the natural transformation* α.

So, we can certainly speak, as before, of the "equality" of categories. We can also speak of the *isomorphism of categories*: an isomorphism between $\mathbf{C}$ and $\mathbf{D}$ is a functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ for which there is an inverse functor $\mathcal{G} : \mathbf{D} \to \mathbf{C}$. I.e., $\mathcal{F}\mathcal{G}$ is the identity functor on $\mathbf{C}$ and $\mathcal{G}\mathcal{F}$ the identity on $\mathbf{D}$, where we define the composition of functors in the obvious way. But because we also have natural transformations, we can also define a subtler notion, the *equivalence of categories.* An equivalence is a functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ together with a functor $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $a : \mathcal{F}\mathcal{G} \to 1_{\mathbf{C}}$

and $b : \mathcal{GF} \to 1_{\mathbf{D}}$. A *natural isomorphism* is a natural transformation which has an inverse.

As we can "relax" the notion of equality to the notion of isomorphism when we pass from sets to categories, we can relax the condition that $\mathcal{FG}$ and $\mathcal{GF}$ equal identity functors to the condition that they be isomorphic to the identity functor when we pass from categories to the 2-category **Cat**.

We need to have the natural transformations to be able to speak of functors being isomorphic, just as we needed functions to be able to speak of sets being isomorphic. In fact, with each extra level in the theory of n-categories, we will be able to come up with a still more refined notion of "n-equivalence" in this way.

Thus, in contrast to a set, which consists of a static collection of "things", a category consists not only of objects or "things" but also morphisms which can viewed as "processes" transforming one thing into another. Similarly, in a 2-category, the 2-morphisms can be regarded as "processes between processes", and so on. The eventual goal of basing mathematics upon omega-categories is thus to allow us the freedom to think of any process as the sort of thing higher-level processes can go between. By the way, it should also be very interesting to consider "$\mathbb{Z}$-categories" (where $\mathbb{Z}$ denotes the integers), having j-morphisms not only for $j = 0, 1, 2, ...$ but also for negative j. Then we may also think of any thing as a kind of process.

2.1 Santilli iso-functors

Santilli's central idea [13] is the generalization of the fundamental unit of the number theory, from its trivial n-dimensional form $I = \text{diag}(1, 1, \ldots)$ to an n-dimensional matrix $\hat{I}$ with the general dependence of all essential variables

$$I = \text{diag}(1, 1, \ldots) \Rightarrow \hat{I} = \hat{I}(s, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \ldots) \qquad (10)$$

under the condition of preserving the original axioms of the unit (nondegeneracy, hermiticity, and positive-definiteness).

The "lifting" $I \Rightarrow \hat{I}$ requires, naturally, for necessary compatibility, a generalization of the conventional associative multiplication $x \circ y$ into the so-called isomultiplication

$$x \circ y \Rightarrow x \,\hat{\circ}\, y := xTy\,, \quad T = \text{fixed}\,, \tag{11}$$

where the quantity T is called the *isotopic* element. Then $\hat{I} = T^{-1}$ is a correct left and right unit element of the theory with respect the new multiplication $\hat{\circ}$ and it is called the *isounit*. On the other hand, R. M. Santilli initiated the functional isoanalysis with the isotopies of basic functions such as exponential, logarithm, trigonometric and other functions, see Chapter 6 of the monograph [13]. The construction of the new isoanalysis was continued by the physicist J. V. Kadeisvili, e.g. in the papers J. V. Kadeisvili [14,15]. Additional work was done by A. K. Aringazin et al, see Appendix 4.B of the monograph R. M. Santilli [16].

DEFINITION 2.3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories. A* **Santilli iso-functor** *from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is a family of functions* $\hat{I}$ *which associates to each object* A *in* $\mathbf{X}$ *an object* $\hat{I}A$ *in* $\mathbf{Y}$ *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A,B)$ *a morphism* $\hat{I}f \in \mathrm{Hom}_{\mathbf{Y}}(\hat{I}A, \hat{I}B)$, *and which is such that:*

(FI) $\hat{I}(g \circ f) = \hat{I}g \circ \hat{I}f$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A,B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B,C)$;

(F2) $\hat{I}\,\mathrm{id}_A = \mathrm{id}_{\hat{I}A}$ *for all* $A \in \mathrm{Ob}\,(\mathbf{X})$.

3 Categories of Groups and iso-Groups

One of the simplest algebraic structures is the structure of a group [17]. A set G of elements of any kind is said to be a *group* if a group operation $a \circ b$ is defined in it satisfying the following axioms:

$G.1^\circ$ For any two elements a and b there exists an element

$$c = a \circ b \tag{12}$$

$G.2^\circ$ This operation is associative, that is, for any three elements a, b, c,

$$(a \circ b) \circ c = a \circ (b \circ c). \tag{13}$$

$G.3°$ There exists a *neutral element* e, i. e. an element such that for every element a,

$$a \circ e = e \circ a = a. \tag{14}$$

$G.4°$ For each element a there exists a *symmetric element* $\bar{a}$ such that

$$a \circ \bar{a} = \bar{a} \circ a = e. \tag{15}$$

If the group operation $a \circ b$ is called *addition*, we write $c = a + b$ and the element c is called the *sum*, the neutral element is called *zero* and is written as 0, the symmetric element is called the *opposite* and is written as $-a$, and the group is called *additive.*

If the group operation $a \circ b$ is called *multiplication*, we write $c = a \cdot b$, or $c = ab$, the element c is called the *product*, the neutral element is called *unit* and is written as 1, the symmetric element is called the *inverse* and is written as a^{-1}, and the group is called *multiplicative.*

If the group satisfies in addition the axiom

$G.5°$. For any two elements a and b

$$a \circ b = b \circ a, \tag{16}$$

then the group is called *commutative* or *Abelian.*

A set of elements endowed with an operation $a \circ b$ without the properties $G.2°$, $G.3°$ and $G.4°$ is called a *magma.* A magma with the property $G.3°$ is called a *unital magma*, a magma with the property $G.2°$ is called a *semigroup.* A magma in which the equations $a \circ x = b$ and $x \circ a = b$ are solvable for all a and b is called a *quasigroup.* A unital semigroup is called a *monoid*, a unital quasigroup is called a *loop.* All these structures (as also the ones to be introduced yet) are termed infinite, respectively finite, if the underlying set is infinite, respectively finite.

DEFINITION 3.1. *A map $f : G \longrightarrow G'$ between two groups $(G, \circ)$ and $(G', \square)$ is called* **homomorphism**, *if the following property holds:*

$$(\forall a, b \in G)[f(a \circ b) = f(a) \square f(b)] \tag{17}$$

Thus a homomorphism "carries" the composition law $\circ$ on G to the composition law $\square$ on G'. Homomorphisms of groups are well visualized in some important aspects with the help of two concepts, the image Im(f) and the kernel Ker(f) of the homeomorphism.

DEFINITION 3.2. *If $f : G \longrightarrow G'$ is a group homomorphism, then we define:*

$$a)\mathrm{Im(f)} = f(a)/a \in G \tag{18}$$

$$b)\mathrm{Ker(f)} = a \in G/f(a) = e' \in G'. \tag{19}$$

It is well known that Im(f) is a subgroup of G' and Ker(f) is a subgroup of G.

DEFINITION 3.3. *A homomorphism f between two groups G and G' is called* **isomorphism** *if f is bijective. In the case where $G = G'$ an homomorphism f is called* **endomorphism** *and an isomorphism is called* **automorphism**.

DEFINITION 3.4. **Grp** *is the category with groups as objects and homomorphisms as morphisms.*

DEFINITION 3.5. *Let* **Grp** *and* **IsoGrp** *be two categories.*
A Santilli iso-functor $\hat{I}$ from **Grp** *associates to each object A in* **Grp** *category an object $\hat{I}A$ in* **IsoGrp** *and to each morphism $f \in \mathrm{Hom}_{\mathbf{Grp}}(A, B)$ a morphism $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoGrp}}(\hat{I}A, \hat{I}B)$, i.e., we reconstruct the elements of the* **category IsoGrp** *as*

$$\mathbf{Grp} \ni A \longrightarrow \hat{A} \equiv A\hat{I} \in \mathbf{IsoGrp}, \tag{20}$$

where the isounit $\hat{I}$ is defined with the help of an invertible element [13]

$$T : \hat{I} = T^{-1}, \tag{21}$$

called isotopic element, and the new composition law is defined by

$$(\forall \hat{A}, \hat{B} \in \mathbf{IsoGrp})[\hat{A}\hat{\circ}\hat{B} \equiv \hat{A}T\hat{B}]. \tag{22}$$

It can be proved easily, that **IsoGrp** for the fixed isotopic element T, with the above internal composition, can become a iso-group $\hat{G}$ with unit $\hat{I}$ [13]. The notions of isomorphism, iso-isomorphism, etc, can be defined in a similar way as above.

DEFINITION 3.6. *Let* $\hat{\mathcal{I}} : \mathbf{Grp} \to \mathbf{IsoGrp}$ *and* $\hat{\mathcal{I}}' : \mathbf{Grp} \to \mathbf{IsoGrp}$ *be two functors. A* **natural transformation** $\alpha : \hat{\mathcal{I}} \to \hat{\mathcal{I}}'$ *is given by the following data:*

For every object A *in* **Grp** *there is a morphism* $\alpha_A : \hat{\mathcal{I}}(A) \to \hat{\mathcal{I}}'(A)$ *in* **IsoGrp** *such that for every morphism* $f : A \to B$ *in* **Grp** *the following diagram is commutative*

$$\begin{array}{ccc} \hat{\mathcal{I}}(A) & \xrightarrow{\alpha_A} & \hat{\mathcal{I}}'(A) \\ \hat{\mathcal{I}}(f) \downarrow & & \downarrow \hat{\mathcal{I}}'(f) \\ \hat{\mathcal{I}}(B) & \xrightarrow{\alpha_B} & \hat{\mathcal{I}}'(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $\hat{\mathcal{I}}(f) \circ \alpha_A = \alpha_B \circ \hat{\mathcal{I}}'(f)$.

The morphisms α_A, $A \in \mathrm{Ob}(\mathbf{Grp})$, are called the *components of the natural transformation* α.

4 Categories of iso-rings and iso-fields

4.1 IsoRng category

Groups are algebraic systems with one internal composition law. More complicated (and hence, richer) systems are obtained if we introduce a second internal composition law, which is related to the first.

If, in a set of elements of any kind two operations $a+b$ and ab are defined such that

$R.l^\circ$ The set is a commutative group with respect to the operation $a+b$;

$A.2^\circ$ The set is a semigroup with respect to the operation ab;

$R.3^\circ$ The operation ab is distributive with respect to the operation $a+b$:

$$a(b+c) = ab + ac, \qquad (a+b)c = ac + bc, \tag{23}$$

the set is called a *ring*.

The ring $\mathbb{Z}$ of integers is a ring with 1, the rings $2\mathbb{Z}$ of even integers, $3\mathbb{Z}$ of integers of the form $3n$, n from $\mathbb{Z}$, and so on, are rings without 1.

DEFINITION 4.1. *A map $f : R \longrightarrow R'$ between two rings $R(+, \star)$ and $R'(\square, \circ)$ is called* **ring homomorphism**, *if it preserves both composition laws, i.e, if the following property holds:*

$$(\forall a, b \in R)[f(a+b) = f(a) \square f(b) \wedge f(a \star b) = f(a) \circ f(b)] \tag{24}$$

DEFINITION 4.2. *A homomorphism f between two rings R and R' is called* **isomorphism** *if f is bijective. In the case where $R = R'$, a homomorphism f is called* **endomorphism** *and an isomorphism is called* **automorphism**.

DEFINITION 4.3. *If $f : R \longrightarrow R'$ is a ring homomorphism, then we define*

$$i)\ Im(f) = f(a) \ / \ a \in R, \tag{25}$$

$$ii)\ Ker(f) = a \in R \ / \ f(a) = e' \in R' \tag{26}$$

where e' is the identity element of the ring R'.

DEFINITION 4.4. **Rng** *is the category with rings as objects and ring homomorphisms as morphisms.*

DEFINITION 4.5. *Let* **Rng** *and* **IsoRng** *be two categories. A Santilli isofunctor $\hat{\mathcal{I}}$ from* **Rng** *associates to each object A in* **Rng** *category an object $\hat{I}A$ in* **IsoRng** *and to each morphism $f \in \mathrm{Hom}_{\mathbf{Rng}}(A, B)$ a morphism $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoRng}}(\hat{I}A, \hat{I}B)$, i.e., we reconstruct the elements of the* **category IsoRng** *as*

$$\mathbf{Rng} \ni A \longrightarrow \hat{A} \equiv A\hat{I} \in \mathbf{IsoRng}, \tag{27}$$

where the isounit $\hat{I}$ is defined with the help of an isotopic element [13]

$$T : \hat{I} = T^{-1}, \tag{28}$$

the first composition law (which is often called addition) is kept unchanged and the second composition law (which is often called multiplication) is defined by

$$(\forall\, \hat{A}, \hat{B} \in \mathbf{IsoRng})[\hat{A}\hat{\star}\hat{B} \equiv \hat{A}T\hat{B}]. \tag{29}$$

It can be proved easily, that **IsoRng** for the fixed isotopic element T, with the above internal composition, can become a iso-ring $\hat{R}$ with isounit $\hat{I}$ [13]. The notions of isomorphism, iso-isomorphism, etc, between two iso-rings can be defined in a similar way as above.

4.2 IsoField category

A ring in which the set of elements without 0 is a commutative group with respect to the operation *ab* is called a *field.*

DEFINITION 4.6. *A* **field** $\mathbb{F}$ *is a commutative ring with unit where every element (except zero), is invertible. More precisely a field $\mathbb{F}(+,\cdot)$ is:*
1) An abelian group with respect to an internal operation, which is usally denoted with + and called addition and
2) is equipped with a second internal operation, denoted with $\cdot$ and called multiplication, so that the following rules hold

$$i)\ (\forall\ \alpha,\beta,\gamma \in \mathbb{F})\ \ [\alpha(\beta\gamma) = (\alpha\beta)\gamma], \tag{30}$$

$$ii)\ (\exists\ 1 \in \mathbb{F})\ (\forall\ \alpha \in \mathbb{F})\ \ [\alpha 1 = 1\alpha = \alpha], \tag{31}$$

$$iii)\ (\forall\ \alpha \in \mathbb{F})\ \ (\exists\ \alpha^{-1} \in \mathbb{F})\ \ [\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1], \tag{32}$$

$$iv)\ (\forall\ \alpha,\beta \in \mathbb{F})\ \ [\alpha\beta = \beta\alpha], \tag{33}$$

$$v)\ (\forall\ \alpha,\beta,\gamma \in \mathbb{F})\ \ [\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma]. \tag{34}$$

A ring in which the set of elements without 0 is a noncommutative group with respect to the operation *ab* is called a *skew field.*

DEFINITION 4.7. *A map* $f : \mathbb{F} \longrightarrow \mathbb{F}'$ *between two fields* $\mathbb{F}(+,\cdot)$ *and* $\mathbb{F}'(\square,\circ)$ *is called* **field homomorphism**, *if it preserves both composition laws, i.e, if the following property holds:*

$$(\forall a, b \in \mathbb{F})[f(a+b) = f(a)\square f(b) \wedge f(a\cdot b) = f(a)\circ f(b)] \tag{35}$$

In the following we shall denote a field for brevity with $\mathbb{F}$ instead of $\mathbb{F}(+,\cdot)$.

The ring $\mathbb{Q}$ of rational numbers, the ring $\mathbb{R}$ of real numbers, the ring $\mathbb{C}$ of complex numbers, the ring $\mathbb{F}_p$ of residues modulo a prime integer p, and the *Galois field* $\mathbb{F}_q$ $(q = p^k)$ obtained from $\mathbb{F}_p$ by adjoining the roots of an irreducible algebraic equation of degree k over $\mathbb{F}_p$, are fields. The ring $\mathbb{H}$ of *quaternions* is a skew field. The fields $\mathbb{F}_p$ and $\mathbb{F}_q$ are finite and contain p and q= p^k elements, respectively. The rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$ are infinite, and the three last fields are *continuous* (*topological*). A field $\mathbb{Q}(\alpha, \beta, \ldots)$ of algebraic numbers, obtained from $\mathbb{Q}$ by adjoining the roots $\alpha, \beta, \ldots$ of an irreducible algebraic equation over $\mathbb{Q}$, and the field $\mathbb{Q}_p$, of *p-adic numbers*, which is also an extension of $\mathbb{Q}$, are also infinite, and the last field is continuous.

DEFINITION 4.8. **Field** *is the category with fields as objects and field homomorphisms as morphisms.*

DEFINITION 4.9. *Let* **Field** *and* **IsoField** *be two categories. A Santilli isofunctor* $\hat{\mathcal{I}}$ *from* **Field** *associates to each object* A *in* **Field** *category an object* $\hat{I}A$ *in* **IsoField** *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{Field}}(A, B)$ *a morphism* $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoField}}(\hat{I}A, \hat{I}B)$, *i.e., we reconstruct the elements of the* **category IsoField** *as*

$$\mathbf{Field} \ni A \longrightarrow \hat{A} \equiv A\hat{I} \in \mathbf{IsoField}, \tag{36}$$

where the isounit $\hat{I}$ *is defined with the help of an isotopic element [13]*

$$T : \hat{I} = T^{-1}, \tag{37}$$

the first composition law (which is often called addition) is kept unchanged and the second composition law (which is often called multiplication) is defined by

$$(\forall \hat{A}, \hat{B} \in \mathbf{IsoField})[\hat{A}\hat{\cdot}\hat{B} \equiv \hat{A}T\hat{B}]. \tag{38}$$

5 Subcategories, subisogroups, subisorings

Subsets of categories, isogroups and isoringsthat are also categories, isogroups and isorings respectively, are called *subcategories*, *subisogroups* and *subisorings*, respectively.

DEFINITION 5.1. *A category* $\mathbf{D}$ *is called a* **subcategory** *of a category* $\mathbf{C}$ *if* $\mathrm{Ob}(\mathbf{D}) \subseteq \mathrm{Ob}(\mathbf{C})$, $\mathrm{Mor}(\mathbf{D}) \subseteq \mathrm{Mor}(\mathbf{C})$, *and morphism composition in* $\mathbf{D}$ *coincide with their composition in* $\mathbf{C}$.

A bijection between two sets endowed with algebraic structures which preserves the operations of these structures is called an *isomorphism*, sets between which there is an isomorphism are called *isomorphic*.

A surjection or injection between two sets endowed with algebraic structures which preserves the operations is called a *homomorphism*, sets between which there is a homomorphism are called *homomorphic*.

If two isogroups $\hat{G}$ and $\hat{H}$ are homomorphic, the elements of $\hat{G}$ corresponding to the neutral element of $\hat{H}$ form a subisogroup $\hat{N}$ of $\hat{G}$, such a subisogroup is called an *invariant subisogroup* or *normal subisogroup*. In this case the isogroup isomorphic to $\hat{H}$ is called the *quotient isogroup* and is written as $\hat{G}/\hat{N}$.

If two isorings $\hat{\mathcal{R}}$ and $\hat{\mathcal{S}}$ are homomorphic, the elements of $\hat{\mathcal{R}}$ corresponding to the zero in $\hat{\mathcal{S}}$ form a subisoring $\hat{\mathcal{J}}$ of $\hat{\mathcal{R}}$ called an *ideal*. In this case the isoring isomorphic to $\hat{\mathcal{S}}$ is called the *quotient isoring* and is written as $\hat{\mathcal{R}}/\hat{\mathcal{J}}$.

Isomorphisms of a isogroup or a isoring onto itself are called *isogroup* or *isoring automorphisms*, respectively; homomorphisms of a isogroup or a isoring into itself are called *isogroup* or *isoring endomorphisms*, respectively.

6 Direct sums and products

If $\hat{G}$ and $\hat{H}$ are two isogroups with operations $\hat{g}_1\hat{\circ}\hat{g}_2$ and $\hat{h}_1\hat{\circ}\hat{h}_2$, the pairs $(\hat{g},\hat{h})$ with operation $(\hat{g}_1,\hat{h}_1)\hat{\circ}(\hat{g}_2,\hat{h}_2) = (\hat{g}_1\hat{\circ}\hat{g}_2, \hat{h}_1\hat{\circ}\hat{h}_2)$ form a isogroup. If $\hat{G}$ and $\hat{H}$ are both additive, respectively multiplicative, isogroups, this isogroup is called a *direct sum*, respectively *direct product*, and is written as $\hat{G}\hat{\oplus}\hat{H}$, respectively $\hat{G}\hat{\otimes}\hat{H}$.

The direct sum $\hat{\mathcal{R}}\hat{\oplus}\hat{\mathcal{S}}$ of rings $\hat{\mathcal{R}}$ and $\hat{\mathcal{S}}$ is similarly defined.

7 Vector isospaces

A set L^n of elements of any kind, called *vectors*, is said to be an *n-dimensional vector space* if in this set the operations of *addition* and of *multiplication by scalars*, that is real numbers, are defined, satisfying:

$VI.1°$ - $5°$ Addition of vectors satisfies axioms $G.1-5°$ for a commutative group;

$VII.1°$ For any vector a and any scalar λ there exists a vector

$$\mathbf{b} = \mathbf{a}\cdot\lambda = \mathbf{a}\lambda \tag{39}$$

called the product of $\mathbf{a}$ by λ;

$VII.2°$ Multiplication by 1 does not change a vector:

$$\mathbf{a}\cdot 1 = \mathbf{a}; \tag{40}$$

$VII.3°$ Multiplication of vectors by scalars is distributive with respect to addition of scalars:

$$\mathbf{a}(\lambda+\mu) = \mathbf{a}\lambda + \mathbf{a}\mu; \tag{41}$$

$VII.4°$ Multiplication of vectors by scalars is distributive with respect to addition of vectors:

$$(\mathbf{a}+\mathbf{b})\lambda = \mathbf{a}\lambda + \mathbf{b}\lambda; \tag{42}$$

$VII.5°$ Multiplication of vectors by scalars is associative:

$$(\mathbf{a}\lambda)\mu = \mathbf{a}(\lambda\mu); \tag{43}$$

and *axioms VIII.*1° - 2° *of dimension*, which are based on the notions of linear independence and dependence of vectors. Vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are said to be *linearly independent* if a linear combination $\mathbf{a}_1\lambda_1 + \mathbf{a}_2\lambda_2 + \ldots + \mathbf{a}_m\lambda_m$ is equal to zero only if all coefficients $\lambda_i = 0$, and *linearly dependent* if there are nonzero coefficients λ_i such that this linear combination is equal to zero.

*VIII.*1° There exist n linearly independent vectors;

*VIII.*2° Any $n + 1$ vectors are linearly dependent.

If we have chosen n linearly independent vectors $\mathbf{e}_1\, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in L_n, then each vector can be written as

$$\mathbf{x} = \sum_i \mathbf{e}_i x^i = \mathbf{e}_i x^i. \tag{44}$$

The numbers x^i are called the *coordinates* of the vector $\mathbf{x}$, the vectors $\mathbf{e}_i$ are called *basis vectors*. Later we will write the sums (44) only in the last form and when in our formulas the same upper and lower indices appear we will always mean summation with respect to these indices.

DEFINITION 7.1. *A subset U of a vector space V is called* **vector subspace** *if it is a subsystem which obeys the axioms of vector space in itself, that is U is closed under vector addition and scalar multiplication.*

DEFINITION 7.2. *The notions of* Ker(f) *and* Im(f) *are defined by the relations*

$$a)\ \mathrm{Ker(f)} = [x \in V\ /\ f(x) = 0 \in U], \tag{45}$$

$$b)\ \mathrm{Im(f)} = [f(x) \in U\ /\ x \in V]. \tag{46}$$

It easy proved that Ker(f) and Im(f) are subspaces of V and U respectively.

DEFINITION 7.3. *Let V and U two vector spaces over the same field $\mathbb{F}$ (not necessarily of the same dimension). A map $f : V \longrightarrow U$ is called* **linear map** *or* **linear transformation** *if the following property is holds:*

$$(\forall \alpha, \beta \in \mathbb{F})\,(\forall x, y \in V)\,[f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)]. \tag{47}$$

In case $V = U$, the map f is called **linear operator**.

Two vector spaces are called *isomorphic* if there is a bijection between them that preserves addition of vectors and multiplication of vectors by scalars, and *homomorphic* if there is a surjection or an injection between them that preserves these operations. Isomorphisms of a vector space onto itself and homomorphisms of a vector space into itself are called *automorphisms* and *endomorphisms* of this vector space respectively. Automorphisms and endomorphisms of a vector space are called *linear transformations* in it.

DEFINITION 7.4. **$\mathbf{Vect}_k$ category** *consisting of vector spaces over a field k as objects and k-linear maps as morphisms;*

From the definition of vector space one can see that we cannot construct an isotopy of a vector space without first introducing an isotopy of the field, because the multiplicative unit I of the space is that of the underlying field. Note that we are lifting of the field, but the elements of the linear space remain unchanged.

DEFINITION 7.5. *Let* **Vect** *and* **IsoVect** *be two categories, V be a vector space over the field $\mathbb{F}$ and $\hat{\mathbb{F}}$ an isofield of $\mathbb{F}$. A Santilli iso-functor $\hat{\mathcal{I}}$ from* **Vect** *associates to each object A in* **Vect** *category an object $\hat{I}A$ in* **IsoVect category** *by "isovector space" as the vector space $\hat{V}$, (which is the same set as V), over the isofield $\hat{\mathbb{F}}$ equipped with a new external operation $\diamond$ which is such to verify all the axioms for a vector space, i.e,*

$$(\forall\ \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ x \in \hat{V})[\hat{\alpha} \diamond (\hat{\beta} \diamond x) = (\hat{\alpha} \star \hat{\beta}) \diamond x], \tag{48}$$

$$(\forall\ \hat{\alpha}, \in \hat{\mathbb{F}})\ \ (\forall\ x, y \in \hat{V})[\hat{\alpha} \diamond (x + y) = (\hat{\alpha} \diamond x + \hat{\beta}) \diamond y], \tag{49}$$

$$(\forall\ \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ x \in \hat{V})[(\hat{\alpha} + \hat{\beta}) \diamond x) = (\hat{\alpha} \diamond x + \hat{\beta}) \diamond x], \tag{50}$$

$$(\forall\ x \in V)[\hat{I} \diamond x = x \diamond \hat{I} = x]. \tag{51}$$

and to each morphism $f \in \mathrm{Hom}_{\mathbf{Vect}}(A, B)$ an isolinear transformation as a morphism $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoVect}}(\hat{I}A, \hat{I}B)$, which is an isomap:

$$\hat{f} : \hat{V} \longrightarrow \hat{V}', \tag{52}$$

between two isolinear vector spaces $\hat{V}$ *and* $\hat{V}'$ *over the same isofield* $\hat{\mathbb{F}}$ *which preserves the sum and isomultiplication, i.e., which is such that*

$$(\forall\ \hat{\alpha},,\hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ x, y \in V)[\hat{f}(\hat{\alpha} \star x + \hat{\beta} \star y) = \hat{\alpha} \star \hat{f}(x) + \hat{\beta} \star \hat{f}(y)]. \quad (53)$$

8 Algebras

If the set A is simultaneously a vector space L^n and a ring $\mathcal{R}$ or a field $\mathbb{F}$ and if the following axiom holds:

$A.1^\circ$ For any two elements a and b and any two scalars λ and μ we have

$$(a\lambda)(b\mu) = (ab)(\lambda\mu), \quad (54)$$

then the set A is called an *n-algebra.*

The product of two basis elements of an algebra can be written as

$$e_i e_j = C_{ij}^k e_k. \quad (55)$$

The numbers C_{ij}^k are called the *structure constants* of the algebra. In this book we will consider both associative and nonassociative algebras. The field $\mathbb{C}$ of complex numbers is a commutative associative algebra with basis $1, i$ $(i^2 = -1)$, the skew field $\mathbb{H}$ of quaternions is a noncommutative associative algebra with basis $1, i, j, k$ $(i^2 = j^2 = k^2 = -l,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j)$.

An algebra which is a field or a skew field is called a *division algebra.*

Two algebras are called *isomorphic* if there is a bijection between them that preserves addition and multiplication of vectors, and multiplication of vectors by scalars, and *homomorphic* if there is a surjection or an injection between them that preserves these operations. Isomorphisms of an algebra onto itself and homomorphisms of an algebra into itself are *algebra automorphisms* and *algebra endomorphisms* of this algebra, respectively.

DEFINITION 8.1. *A subset* V *of an algebra* A *is called a subalgebra if it satisfies the algebra axioms, i.e., if it is an algebra in its own right.*

It is easy to proove that a subset $\mathsf{V} \subset \mathsf{A}$ of an algebra A is a subalgebra iff:

$$(\forall x, y \in \mathsf{A})(\forall \alpha \in \mathbb{F})[x + y \in \mathsf{A}, \alpha x \in \mathsf{A}, xy \in \mathsf{A}] \tag{56}$$

that is, the subset V is closed under all the composition laws.

DEFINITION 8.2. *A map* $f : \mathsf{A} \longrightarrow \mathsf{A}'$ *between algebras over the same field* $\mathbb{F}$ *is called* **morphism** *if:*

$$(\forall x, y \in \mathsf{A})(\forall \alpha, \beta \in \mathbb{F})[f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \ \wedge \ f(xy) = f(x)f(y)]. \tag{57}$$

DEFINITION 8.3. *The usual statements about the image and kernel of a morphisms (57) hold.*

DEFINITION 8.4. **Alg** *is the category with algebras as objects and maps (57) as morphisms.*

DEFINITION 8.5. *Let* **Alg** *and* **IsoAlg** *be two categories.*
A Santilli iso-functor $\hat{\mathcal{I}}$ *from* **Alg** *associates to each object* A *in* **Alg** *category an object* $\hat{\mathsf{A}}$ *in* **IsoAlg** *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{Alg}}(\mathsf{A}, \mathsf{B})$ *a morphism* $\hat{f} \in \mathrm{Hom}_{\mathbf{IsoAlg}}(\hat{\mathsf{A}}, \hat{\mathsf{B}})$, *i.e., we reconstruct the elements of the* **category IsoAlg** *as an "isotope"* $\hat{\mathsf{A}}$ *of an algebra* A *with elements* $x, y, z, \cdots$ *and product* xy *over a field* $\mathbb{F}$, *is the same vector space* L^n *but defined over the isofield* $\hat{\mathbb{F}}$, *equipped with a new product* $x \star y$, *called "isotopic product", which is such to verify the original axioms of* A.

Acknowledgments

The author is grateful to Prof. C. Corda as a referee for helpful discussions and useful recommendations leading to a better structure of the present article.

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THE HORIZONTAL LIFTS TO THE EXTENDED ACCELERATION BUNDLE

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Received February 24, 2014

Abstract

The aim of this study is to obtain the restrictions to the extended acceleration bundle of the horizontal lifts to double tangent bundle of the basic tensor fields obtained in terms of vertical, complete and horizontal lifts on the tangent bundle by using the Sasaki Riemann metric on the tangent bundle.

Keywords: Sub-bundle of Tangent Bundle, Lifting Theory, Sasaki Riemann metric on Tangent Bundle

AMS Classification: 58A30, 28A51, 53C07

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1. Introduction

In classic mechanic, geodesics can be considered as trajectories of particles in a system with n particle moving by the effects of the gravitational fields. To obtain the more knowledge about the motion of particles in system, the position, velocity and acceleration of each particle in system at any instant t must be known.

The extended acceleration bundle of a manifold M provides a geometric approach to these mechanic problems. Assuming that $\mathfrak{I}$ is one parameter curves space on M which configuration space of system with n particle. If the positions, velocities and accelerations at initial point of the curves in $\mathfrak{I}$ are the same, it can be said that these curves whose equivalent to one another. This equivalent relation produces a equivalence classes in $\mathfrak{I}$. In this paper, we assume that the representative curve of each the equivalence class is a geodesic of M. The second order ordinary differential equations on M whose particular solutions are geodesics of M are represented by a vector field on TM, which is called spray. Spray is a horizontal vector field on TM. Moreover, spray defines horizontal curves in the extended acceleration bundle of M with 3n dimensional differentiable manifold structure.

The aim of this article is to find that lifts of the basic differential geometric objects from a manifold to its the extended acceleration bundle.

The lifts from a manifold to its second order tangent bundle or its double tangent bundle are found in literature.
The vertical and complete lifts from a manifold to its second order extended manifold (extended acceleration bundle) are also found.
Moreover the horizontal lifts of some tensor fields from a manifold to its second order extended manifold (the extended acceleration bundle) by using the metric on TM obtained by complete lift of a (pseudo) Riemann metric of M has been considered.
So far the horizontal lifts of some tensor fields from a manifold to its the extended acceleration bundle by using the metric on TM obtained by diagonal lift of a Riemann metric of M hasn't been considered.

Yano and Ishihara [12] defined the second order tangent bundle of a manifold and considered new lifts of some tensor fields from a manifold to the second order tangent bundles.
Esin and Civelek [9] and [10] found out the vertical and complete lifts of the basic tensor fields from a manifold to its the double tangent bundle and its second order extended manifold.
Ayhan et al. [4] obtained the horizontal lifts of the basic tensor fields from a manifold to its the double tangent bundle by using the Sasaki Riemann metric on the tangent bundle.
Aycan [3] considered the vertical and complete lifts of the basic tensor fields from a manifold to its the higher order extension
Civelek et al. [7] studied on the horizontal lifts of some tensor fields from a manifold to its second order extended manifold by using the metric on TM obtained by complete lift of a (pseudo) Riemann metric of M

In this study, we studied on the restriction of the horizontal lifts of the basic tensor field in tangent bundle TM of a manifold M obtained in terms of the vertical, complete and horizontal lifts to the extended acceleration manifold ${}^2M$ by using the Sasaki Riemann metric g^S on TM.

2. The Extended Acceleration Bundles

In this section, we will consider the concepts as the double tangent bundle, the second order tangent bundle or the extended acceleration bundle. Moreover, we will show that the extended acceleration bundle of a n-dimensional manifold is a 3n-dimensional manifold which is consisted of sprays.

Definition 1 Let M be a n-dimensional differentiable manifold and $(x^1, x^2, ..., x^n)$ be the local coordinates at a point $p \in U \subset M$. T_pM is called as tangent vector space at a point p of M and $(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$ is called as a frame at a point p on M. T_p^*M is called co-tangent vector

space at a point p of M and $(dx^1,...,dx^n)$ is called as a dual frame a point p on M.

Definition 2 The disjoint union of each tangent vector space at all point of M is called as the tangent bundle of M denoted by

$$TM = \bigcup_{\forall p \in M} T_p M. \tag{1}$$

Any point in TM is represented by (p, X) where p is a point in M and X is a tangent vector at a point p of M. $\pi_1 : TM \to M$ is called as canonical projection map. Let $\tilde{p} = (p, X)$ be a point in $\pi_1^{-1}(U) \subset TM$. Then $\left(x^1,...,x^n, y^1,...,y^n\right)$ is a the induced local coordinates of a point $\tilde{p}$ in TM where $\{x^i\}, i = 1,...,n$ is the local components of a point $\pi_1(\tilde{p}) = p$ of M and $\{y^i\}, i = 1,...,n$ is the local components of the tangent vector X providing $y^i = dx^i(X) = X[x^i]$. Thus TM has 2n dimensional manifold structure [13].

Definition 3 The disjoint union of each tangent vector space at all point of TM is called as the tangent bundle of TM or double tangent bundle of M and represented by

$$TTM = \bigcup_{\forall (p,X) \in TM} T_{(p,X)} TM. \tag{2}$$

Any point in TTM is created by $A_{(p,X)}$ where (p, X) is a point in TM and $A_{(p,X)}$ is a tangent vector at a point (p, X). $\pi_2 : TTM \to M$ is called as canonical projection map providing $\pi_2(p, X, A_{(p,X)}) = p$. Moreover, the projection $\pi_{12} : TTM \to TM$ can be defined such that $\pi_2 = \pi_1 \circ \pi_{12}$ and $\pi_{12}(p, X, A_{(p,X)}) = (p, X)$. Let $\tilde{\tilde{p}}$ be a point in $\pi_2^{-1}(U) \subset TTM$. Then $\left(x^1,...,x^n, y^1,...,y^n, z^1,...,z^n, t^1,...,t^n\right)$ is a local coordinates of a point $\tilde{\tilde{p}}$ in TTM where $\{x^i\}, i = 1,...,n$ is the local components of a point $\pi_2(\tilde{\tilde{p}}) = p$, $\{y^i\}, i = 1,...,n$ is the local coordinate function of the tangent vector X_p providing

$y^i = dx^i(X_p) = X_p[x^i]$, $\{z^i\}, i = 1,...,n$ is the local coordinate function of the tangent vector $A_{(p,X)}$ providing $z^i = dx^i(A_{(p,X)}) = A_{(p,X)}[x^i]$ and $\{t^i\}, i = 1,...,n$ is the local coordinate function of the tangent vector $A_{(p,X)}$ providing $t^i = dy^i(A_{(p,X)}) = A_{(p,X)}[y^i]$. Thus TTM has 4n dimensional manifold structure [4, 9].

Theorem 4 TTM, the tangent bundle of TM, has sub-vector bundle $VTM = Ker(\pi_1)_*$ which is called vertical distribution on TM and HTM which is called horizontal distribution on TM. In addition, it can be expressed TTM as direct sum of sub-vector bundles VTM and HTM

$$TTM = VTM \oplus HTM \tag{3}$$

$\{\delta_i, \dot{\partial}_i; 1 \le i \le n\}$ is adapted local frame in TM where δ_i is local frame in HTM

$$\delta_i = \left(\frac{\partial}{\partial x^i}\right)^H = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^k \Gamma_{kj}{}^i \frac{\partial}{\partial y^j}, \tag{4}$$

and $\dot{\partial}_i$ is local frame in VTM

$$\dot{\partial}_i = \left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}. \tag{5}$$

Furthermore, $\{\delta y^i, dx^i; 1 \le i \le n\}$ is adapted local dual frame in TM where

$$\delta y^i = dy^i + y^k \Gamma_{kj}{}^i dx^j, \tag{6}$$

[8].

The Sasaki Riemann metric g^S, the diagonal lift of the Riemann metric g with components g_{ij} on M, has components

$$g^S = g_{ij} dx^i dx^j + g_{ij} \delta y^i \delta y^j \tag{7}$$

with respect to adapted local frame in TM [11].

Theorem 5 The connection coefficients of Levi Civita connection ∇^S of TM with respect to the pseudo-Riemann metric g^S are

$$\nabla^S_{\delta_i}\delta_j = \Gamma^h_{ji}\delta_h + \frac{1}{2}y^l R^h_{jil}\dot{\partial}_h, \quad \nabla^S_{\delta_i}\dot{\partial}_j = \frac{1}{2}y^l R^h_{jil}\delta_h + \Gamma^h_{ji}\dot{\partial}_h,$$
$$\nabla^S_{\dot{\partial}_i}\delta_j = -\frac{1}{2}y^l R^h_{jil}\delta_h, \qquad \nabla^S_{\dot{\partial}_i}\dot{\partial}_j = 0 \tag{8}$$

and the value on dual base one forms of TM of ∇^S are

$$\nabla^S_{\delta_i}dx^j = -\Gamma^h_{ji}dx^h - \frac{1}{2}y^l R^j_{hil}\delta y^h, \quad \nabla^S_{\delta_i}\delta y^j = -\frac{1}{2}y^l R^j_{hil}dx^h - \Gamma^h_{ji}\delta y^h,$$
$$\nabla^S_{\dot{\partial}_i}dx^j = \frac{1}{2}y^l R^j_{hil}dx^h, \qquad \nabla^S_{\dot{\partial}_i}\delta y^j = 0, \tag{9}$$

[13].

Theorem 6 Let f be a differentiable function, $X = X^h\frac{\partial}{\partial x^h}$ be a vector field and $\omega = \omega_i dx^i$ be a one form on M. The vertical and complete lifts of these basic tensor fields have the local expressions

$$i) f^V = fo\tau_M, \qquad f^C = y^i(\frac{\partial f}{\partial x^i})^V, f^H = 0$$
$$ii) X^V = (X^h)^V\frac{\partial}{\partial y^h}, \quad X^C = (X^h)^V\frac{\delta}{\delta x^h} + y^i(\nabla_i X^h)^V\frac{\partial}{\partial y^h}, X^H = (X^h)^V\frac{\delta}{\delta x^h} \tag{10}$$
$$iii)\omega^V = (\omega_i)^V dx^i, \qquad \omega^C = (\omega_i)^V\delta y^i + y^i(\nabla_j\omega_i)^V dx^i, \omega^H = (\omega_i)^V\delta y^i$$

where $\nabla_i X^h = \partial_i X^h + X^k\Gamma^h_{ik}$ and $\nabla_j\omega_i = \partial_j\omega_i - \omega_k\Gamma^k_{ji}$ [13].

Theorem 7 The vertical and complete lifts of these basic tensor fields to double tangent bundle of M have the local expressions as follow:

$$i) \quad \begin{aligned} &f^{VV} = fo\pi_1 o\pi_{12}, \qquad f^{CV} = y^i(\frac{\partial f}{\partial x^i})^{VV}, \\ &f^{VC} = z^i(\frac{\partial f}{\partial x^i})^{VV}, \quad f^{CC} = y^i z^i(\frac{\partial^2 f}{\partial x^i\partial x^j})^{VV} + t^i(\frac{\partial f}{\partial x^i})^{VV}, \end{aligned} \tag{11}$$

$$ii) \quad \begin{aligned} &\left(\frac{\partial}{\partial x^i}\right)^{VV} = \left(\frac{\partial}{\partial y^i}\right)^V = \frac{\partial}{\partial t^i}, \quad \left(\frac{\partial}{\partial x^i}\right)^{VC} = \left(\frac{\partial}{\partial y^i}\right)^C = \frac{\partial}{\partial y^i}, \\ &\left(\frac{\partial}{\partial x^i}\right)^{CV} = \left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial z^i}, \quad \left(\frac{\partial}{\partial x^i}\right)^{CC} = \left(\frac{\partial}{\partial x^i}\right)^C = \frac{\partial}{\partial x^i}, \end{aligned} \tag{12}$$

$$iii)\quad \begin{matrix} \left(dx^i\right)^{VT} = \left(dx^i\right)^{V} = dx^i, & \left(dx^i\right)^{VC} = \left(dx^i\right)^{C} = dz^i, \\ \left(dx^i\right)^{CT} = \left(dy^i\right)^{V} = dy^i, & \left(dx^i\right)^{CC} = \left(dy^i\right)^{C} = dt^i \end{matrix} \tag{13}$$

[9] and [10].

Theorem 8 Let $T_{(p,X,A)}TTM$ be a tangent vector at any point (p,X,A) of TTM. The local frame on the point (p,X,A) of TTM is $\left\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial z^i},\frac{\partial}{\partial t^i}\right\}$ for $i=1,...,n,$ and dual frame on the point (p,X,A) of TTM is $\left\{dx^i,dy^i,dz^i,dt^i\right\}$, with respect to induced coordinates (x^i,y^i,z^i,t^i) $i=1,...,n$ of TTM [4].

Definition 9 Let M be an n-dimensional differentiable manifold and R be real line. If two mappings $F:R\to M$ and $G:R\to M$ satisfy the conditions,

$$F(0)=G(0),\quad \frac{dF^h(0)}{dt}=\frac{dG^h(0)}{dt},\quad \frac{d^2F^h(0)}{dt^2}=\frac{d^2G^h(0)}{dt^2}, \tag{14}$$

the mappings F and G being respectively represented by $x^h=F^h(t)$ and $x^h=G^h(t)$ $\left(t\in R\right)$ with respect to local coordinates $\left(x^h\right)$ defined in a coordinate neighborhood containing the point $F^h(0)=G^h(0)$, then we say that the two mappings F and G are equivalent to each other and write $F\sim G$. Each equivalence class is called as an 2-jet of M. If this class contains a one parameter curve $F:R\to M$ such that $F(0)=P$. The set of all 2-jets of M is denoted T^2M and called as the second order tangent bundle of M. Let (U,x^h) be a coordinate neighbourhood of M. Then 2-jet has the following induced coordinates:

$$x^h=F^h(0),\quad y^h=\frac{dF^h(0)}{dt},\quad z^h=\frac{d^2F^h(0)}{dt^2}, \tag{15}$$

[12]. In the rest of paper, we assume that F(t) is a geodesic of M.

Definition 10 If M is n-dimensional differentiable manifold and its tangent bundle TM must be 2n-dimensional manifold and its double

tangent bundle TTM must be 4n-dimensional manifold. By helping π_1, π_2 and π_{12} maps, following commutative diagram can be define:

$$\begin{array}{ccc} TTM & \xrightarrow{(\pi_1)_*} & TM \\ \pi_{12}\downarrow & & \downarrow\pi_1 \\ TM & \xrightarrow[\pi_1]{} & M \end{array} \qquad (16)$$

If we take

$$^2M = \{A : A \in TTM, \pi_{12}(A) = (\pi_1)_*(A)\} \qquad (17)$$

then ^2M is an embedded sub-manifold of TTM. ^2M is called as the second order extended manifold [5] or the extended acceleration bundle of M [1].

Definition 11 A spray on M is an assignment which gives, for each X in each T_pM a C^r geodesic α_X in M, such that the following hold

1) The domain of each $x^i \circ \alpha_X = \alpha_i$ is an open interval of real numbers containing 0, where x^i is the local coordinate function at a point $p = \alpha_X(0) \in U \subset M$.

2) $(\alpha_X)_*(0) = X$,

3) $\alpha_{(\alpha_X)_*(s)}(t) = \alpha_X(s+t)$,

4) $\alpha_{sX}(t) = \alpha_X(st)$,

5) Let 0 be the subset of $R \times TM$ consisting of all (t, X) for which $\alpha_X(t)$ is defined and F the mapping of 0 into $TM : F(t, X) = (\alpha_X)_*(t)$. Then $F \in C^r$ [2].

Definition 12 There is still another formulation the notation of a spray in terms of infinitesimal generator S of the flow F. If the flow F is given then one defines the vector field S, on TM, by S_X: "The tangent vector to $(\alpha_X)_*$ at $\alpha_X(0)$". This vector field has the property: $(\pi_1)_*(S_X) = X$. [2].

Theorem 13 The local components of the vector field S in TM correspond to a point in TTM are projected to ${}^2M$ with 3n-dimension.

Proof If $X_0 \in T_pM$, then the geodesics $c: t \to c(t)$ passing thorough a point $p = c(0)$ with $\dot{c}(0) = X_0$ are the images under the projection π of the integral curves $X: t \to X(t)$ of the vector field S, which is define a spray for M, i.e. $c = \pi \circ X$ and $X(0) = X_0$. Thus a spray for M is a vector field on TM whose the images under π of integral curves are geodesics of M given by the following second order ordinary differential equations:

$$\left.\frac{d^2(x^k \circ c)}{dt^2}\right|_t = -\Gamma^i_{jk}(c(t))\left.\frac{d(x^j \circ c)}{dt}\right|_t \left.\frac{d(x^k \circ c)}{dt}\right|_t \tag{18}$$

Namely a spray for M is given by

$$S_{X(t)} = y^i(X(t))A_i|_{X(t)} + \dot{y}^i(X(t))A_{n+i}|_{X(t)}, \tag{19}$$

where $y^i(X(t)) = \dfrac{d(x^i \circ c)}{dt}$ and $\dot{y}^i(X(t)) = \left.\dfrac{d^2(x^k \circ c)}{dt^2}\right|_t$.

A tangent vector S_{X_0} at point $X(0)$ of the vector field $S_{X(t)}$ of provides $(\pi_1)_*(S_{X_0}) = X_0$ and S_{X_0} is a horizontal vector in $T_{X_0}TM$ which has the local expression

$$S_{X_0} = y^i(X_0)A_i|_{X_0} - (\Gamma^i_{jk} \circ \pi_1)y^j(X_0)y^k(X_0)A_{n+i}|_{X_0} \tag{20}$$

The induced coordinates of the point S_{X_0} in TTM are

$$(x^1(p),...,x^n(p),\quad X^1_0,...,X^n_0,\quad X^1_0,...,X^n_0,\quad -X^k_0X^j_0\Gamma^1_{jk},...,-X^k_0X^j_0\Gamma^n_{jk}). \tag{21}$$

where $y^i(X_0) = dx^i(X_0) = X_0[x^i] = X^i_0 = \left.\dfrac{d(x^i \circ c)}{dt}\right|_{t=0}$ for $i = 1,...,n$. The projections of the integral curves $X: t \to X(t)$ of the vector field S on TM under the π_1 correspond to geodesics in M [8]. Thus the components corresponding the coordinate functions y^i and z^i of the point (p, X_0, S_{X_0}) in TTM are equal. For that reason TTM can be restricted by 3n-dimensional extended acceleration bundle ${}^2M$.

Theorem 14 Any horizontal vector in the tangent vector space of TM does not correspond to a point in ${}^2M$.

Proof Let $c: t \in I \subset R \to c(t) \in M$ be curve on M and Z be parallel vector field along the curve c(t). Namely $\nabla_X Z = 0$ where $X = \dot{c}(0)$. Then the curve $Z \circ c: t \to Z(c(t))$ defines a curve in TM. Since $Z: M \to TM$ is a transformation that assigns to each point $c(0) = p$ of M is a tangent vector Z_p, $Z_*: TM \to TTM$ must a transformation that assigns to each point $(Z \circ c)(0) = (p, Z)$ of TM is a vector $Z_* X$ expressed by

$$Z_* X = X^i A_i\big|_{(p,Z)} + X^k \frac{\partial Z^i}{\partial x^k} A_{n+i}\big|_{(p,Z)} \in T_{(p,Z)}TM \tag{22}$$

where $A_i\big|_{(p,Z)}$ and $A_{n+i}\big|_{(p,Z)}$ base vectors of $T_{(p,Z)}TM$.

The images of these base vectors under $(\pi_1)_*$ and the map K are

$$(\pi_1)_*(A_i\big|_{(p,Z)}) = \frac{\partial}{\partial x^i}\bigg|_p, \quad (\pi_1)_*(A_{n+i}\big|_{(p,Z)}) = 0_p, \tag{23}$$

$$K(A_i\big|_{(p,Z)}) = (\nabla_{\frac{\partial}{\partial x^i}} Z)_p = 0_p, \quad K(A_{n+i}\big|_{(p,Z)}) = \frac{\partial}{\partial x^i}\bigg|_p. \tag{24}$$

Thus we get

$$(\pi_1)_*(Z_* X) = X^i \frac{\partial}{\partial x^i} = X_p, \quad K(Z_* X) = \nabla_X Z = 0. \tag{25}$$

Since the kernel of K is defined as the horizontal subspace of $T_{(p,Z)}TM$, $Z_* X$ must a horizontal vector in $T_{(p,Z)}TM$. The induced coordinates of the point $(p, Z, Z_* X)$ in TTM are obtained as follow:

$$(x^1(p),...,x^n(p),\ Z^1(p),...,Z^n(p),\ X^1(p),...,X^n(p),\ X^k \frac{\partial Z^1}{\partial x^k},...,X^k \frac{\partial Z^n}{\partial x^k}). \tag{26}$$

These coordinates in (31) see that each horizontal vector in the tangent vector space of TM does not correspond to a point in ${}^2M$ because y^i and z^i coordinates of point $(p, Z, Z_* X)$ in TTM are not equal.

Theorem 15 ${}^2M$ is a 3n-dimensional differentiable manifold.

Proof Let $(U,(x^h))$ and $(U',(x^{h'}))$ be two intersecting coordinate neighbourhoods of M. Let $(\tau^{-1}(U),(x^h,y^h,z^h))$ and $(\tau^{-1}(U'),(x^{h'},y^{h'},z^{h'}))$ be the coordinates induced respectively from $(U,(x^h))$ and $(U',(x^{h'}))$ in ${}^2M$, where $\tau:{}^2M\to M$ is canonical projection. Then denoting by $x^{h'}=x^{h'}(x^i)$ the coordinate transformation in $U\cap U'$, the transformation of the induced coordinates in $\tau^{-1}(U\cap U')$ is given by

$$x^{h'}=x^{h'}(x^i),\quad y^{h'}=\frac{\partial x^{h'}}{\partial x^h}y^h, \tag{27}$$

and

$$\begin{aligned} z^{h'} &= -\Gamma^{h'}_{i'j'}y^{i'}y^{j'} = -\left(\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\Gamma^h_{ij}\frac{\partial x^{h'}}{\partial x^h}+\frac{\partial x^{h'}}{\partial x^m}\frac{\partial^2 x^m}{\partial x^{i'}\partial x^{j'}}\right)\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}y^iy^j \\ &= -\left(\Gamma^h_{ij}\frac{\partial x^{h'}}{\partial x^h}+\frac{\partial x^{h'}}{\partial x^m}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial^2 x^m}{\partial x^{i'}\partial x^{j'}}\right)y^iy^j \\ &= \frac{\partial x^{h'}}{\partial x^h}z^h-\frac{\partial^2 x^{h'}}{\partial x^i\partial x^j}y^iy^j. \end{aligned} \tag{28}$$

Since the coordinate functions (x^h,y^h,z^h) of each point of $\tau^{-1}(U\cap U')\subset{}^2M$ are differentiable, ${}^2M$ must a 3n-dimensional differentiable manifold.

Theorem 16 ${}^2M$ has the local frame expressed by

$$\left\{\frac{\partial}{\partial x^{i'}},\frac{\partial}{\partial y^{i'}},\frac{\partial}{\partial z^{i'}};1\le i\le n\right\}, \tag{29}$$

and the local dual frame expressed by

$$\{dx^{i'},dy^{i'},dz^{i'};1\le i\le n\}. \tag{30}$$

Proof If we take y^i instead of z^i and z^i instead of t^i in (18) and (21), we obtain a local frame and local dual frame at any point of ${}^2M$. Since ${}^2M$ is the sub-manifold with 3n-dimensional of TTM, the claim of this theorem must true.

3. Lifted Functions in Extended Acceleration Bundle

In this section, we will define the horizontal lift of a function $\tilde{f}$ on TM to ${}^2M$. Then we will obtain the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of a function f on M to ${}^2M$ by using the Sasaki Riemann metric on the tangent bundle of M.

Theorem 17 Let $\tilde{f}$ be a function of TM and ∇^S be Levi Civita connection with respect to Sasaki Riemann metric g^S. The horizontal lift of $\tilde{f}$ is

$$\tilde{f}^H = \tilde{f}^C - \nabla^S_{\tilde{\gamma}}\tilde{f} = 0. \tag{31}$$

Proof Since $\nabla^S_{\tilde{\gamma}}\tilde{f} = \tilde{\gamma}(\nabla^S\tilde{f}) = \tilde{\gamma}\left(\left(\nabla^S_{\delta_i}\tilde{f}\right)dx^i + \left(\nabla^S_{\dot{\partial}_i}\tilde{f}\right)\delta y^i\right)$,

$\tilde{f}^C = z^i\partial_i\tilde{f} + t^i\dot{\partial}_i\tilde{f}$ and the linear transformation $\tilde{\gamma}$ is given by

$$\begin{aligned}
\tilde{\gamma}: \quad \Im^1_1(TM) &\rightarrow \Im^1_0(TTM) \\
A = A^i_j dx^j \otimes \frac{\partial}{\partial y^i} &\rightarrow \tilde{\gamma}(A) = z^j A^i_j \left(\frac{\partial}{\partial y^i}\right)^V \\
B = B^i_j dy^j \otimes \frac{\partial}{\partial x^i} &\rightarrow \tilde{\gamma}(B) = t^j B^i_j \left(\frac{\partial}{\partial x^i}\right)^V
\end{aligned} \tag{32}$$

$\tilde{f}^H$ has the local expression

$$\tilde{f}^H = z^i\partial_i\tilde{f} + t^i\dot{\partial}_i\tilde{f} - z^i\partial_i\tilde{f} + y^j z^k\Gamma^i_{kj}\dot{\partial}_i\tilde{f} - t^i\dot{\partial}_i\tilde{f} - y^j z^k\Gamma^i_{jk}\dot{\partial}_i\tilde{f} = 0 \tag{33}$$

Definition 18 Let $\tilde{f}$ be a function of TM. The restriction to ${}^2M$ of the horizontal lift of $\tilde{f}$ has the local expression

$$\tilde{f}^H\Big|_{{}^2M} = 0, \tag{34}$$

and $\widetilde{f}^H\big|_{^2M}$ is called as the horizontal lift of a function $\widetilde{f}$ from TM to ^2M.

Theorem 19 Let f^V, f^C, f^H be vertical, complete and horizontal lift of a function of f on M. Then the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of a function f on M to ^2M are

i) $f^{VH}\big|_{^2M} = 0,$

ii) $f^{CH}\big|_{^2M} = 0,$

iii) $f^{HH}\big|_{^2M} = 0.$

Proof If we put f^V, f^C, f^H instead of $\widetilde{f}$ in (34) we obtain $f^{VH}\big|_{^2M}$, $f^{CH}\big|_{^2M}$ and $f^{HH}\big|_{^2M}$ straightforward.

4. Lifted Vector Fields in Extended Acceleration Bundle

In this section, we will define the horizontal lift of a vector field $\widetilde{X}$ on TM to ^2M. Then we will obtain the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of a vector field X on M to ^2M.

Definition 20 Let M be flat Riemann manifold with the Riemann metric g. $\widetilde{X}$ a vector field on TM and ∇^S Levi Civita connection with respect to g^S Sasaki-Riemann metric. If $\widetilde{X}^H$ provides the following equality:

$$\widetilde{X}^H = \widetilde{X}^C - \nabla^S_{\widetilde{Y}}\widetilde{X} \tag{35}$$

$\widetilde{X}^H$ is called as the horizontal lift of $\widetilde{X}$ to TTM [4].

Theorem 21 The complete lift of the vector field $\widetilde{X}$ in TM given by

$$\widetilde{X} = \widetilde{X}^i\delta_i + \widetilde{X}^{n+i}\dot{\partial}_i,$$

with respect to adapted frame of TM has the following matrix representation:

$$\widetilde{X}^C : \begin{pmatrix} \widetilde{X}^i \\ \widetilde{X}^{n+i} - y^j \widetilde{X}^h \Gamma^i_{jh} \\ z^k \partial_k \widetilde{X}^i + t^k \dot{\partial}_k \widetilde{X}^i \\ z^k \partial_k \left(\widetilde{X}^{n+i} - y^j \widetilde{X}^h \Gamma^i_{jh} \right) + t^k \dot{\partial}_k \left(\widetilde{X}^{n+i} - y^j \widetilde{X}^h \Gamma^i_{jh} \right) \end{pmatrix}, \tag{36}$$

with respect to induced coordinates of TTM [4].

Theorem 22 Let (M, g) flat Riemann manifold and ∇^S Levi Civita connection depended on the Sasaki Riemann metric on TM. For any vector field $\widetilde{X}$ on TM, the vector field $\nabla^S_{\widetilde{Y}} \widetilde{X}$ on TTM has the following matrix representation

$$\nabla^S_{\widetilde{Y}} \widetilde{X} : \begin{pmatrix} 0 \\ 0 \\ z^k \left(\delta_k \widetilde{X}^i + \widetilde{X}^h \Gamma^i_{hk} \right) + \left(t^k + y^j z^k \Gamma^h_{jk} \right) \dot{\partial}_k \widetilde{X}^i \\ \left\{ z^k \left(\delta_k \widetilde{X}^{n+i} + \widetilde{X}^{n+h} \Gamma^i_{hk} \right) + \left(t^k + y^j z^k \Gamma^h_{jk} \right) \dot{\partial}_k \widetilde{X}^{n+i} - \right. \\ \left. - \Gamma^i_{jh} y^j \left[z^k \left(\delta_k \widetilde{X}^i + \widetilde{X}^h \Gamma^i_{hk} \right) + \left(t^k + y^j z^k \Gamma^h_{jk} \right) \dot{\partial}_k \widetilde{X}^i \right] \right\} \end{pmatrix}, \tag{37}$$

with respect to induced coordinates of TTM.

Proof Let (M, g) be flat Riemann manifold and g^S the Sasaki Riemann metric on M. ∇^S Levi Civita connection on TM depended on g^S and $\widetilde{X} = \widetilde{X}^i \delta_i + \widetilde{X}^{n+i} \dot{\partial}_i$ a vector field on TM. Then the vector field $\nabla^S_{\widetilde{Y}} \widetilde{X}$ on TTM is calculated as follow:

$$\begin{aligned} \nabla^S_{\widetilde{Y}} \widetilde{X} &= \gamma(dx^k) \left(\nabla_{\delta_k} \widetilde{X}^i + \widetilde{X}^h \Gamma^i_{kh} \right) (\delta_i)^V + \gamma(\delta y^k) \left(\nabla_{\delta_k} \widetilde{X}^{n+i} + \widetilde{X}^h \Gamma^i_{kh} \right) (\dot{\partial}_i)^V + \\ &+ \gamma(dx^k) \dot{\partial}_k \left(\widetilde{X}^i \right) (\delta_i)^V + \gamma(\delta y^k) \dot{\partial}_k \left(\widetilde{X}^{n+i} \right) (\dot{\partial}_i)^V \\ &= \left[z^k \left(\delta_k \widetilde{X}^i + \widetilde{X}^h \Gamma^i_{hk} \right) + \left(t^k + y^j z^k \Gamma^h_{jk} \right) \dot{\partial}_k \widetilde{X}^i \right] \frac{\partial}{\partial z^i} + \left\{ z^k \left(\delta_k \widetilde{X}^{n+i} + \widetilde{X}^{n+h} \Gamma^i_{hk} \right) + \right. \\ &+ \left. \left(t^k + y^j z^k \Gamma^h_{jk} \right) \dot{\partial}_k \widetilde{X}^{n+i} - \Gamma^i_{jh} y^j \left[z^k \left(\delta_k \widetilde{X}^i + \widetilde{X}^h \Gamma^i_{hk} \right) + \left(t^k + y^j z^k \Gamma^h_{jk} \right) \dot{\partial}_k \widetilde{X}^i \right] \right\} \frac{\partial}{\partial t^i}. \end{aligned}$$

Theorem 23 Let (M, g) be flat Riemann manifold and ∇^S, Levi Civita connection depend on the Sasaki Riemann metric g^S on TM, Since the vertical, complete and horizontal lift of the vector field X in M to TM are X^V, X^C and X^H, the horizontal lifts of X^V, X^C, X^H have the following local matrix representations:

$$i)\quad X^{VH} = \begin{pmatrix} 0 \\ X^i \\ 0 \\ -z^k X^h \Gamma^i_{hk} \end{pmatrix}, \tag{38}$$

$$ii)\quad X^{CH} = \begin{pmatrix} X^i \\ y^j \dfrac{\partial X^i}{\partial x^j} \\ -z^k X^h \Gamma^i_{kh} \\ -y^j z^k \partial_k \left(X^h \Gamma^i_{jh} \right) - t^k X^h \Gamma^i_{kh} \end{pmatrix}, \tag{39}$$

$$iii)\quad X^{HH} = \begin{pmatrix} X^i \\ -y^j X^h \Gamma^i_{jh} \\ -z^k X^h \Gamma^i_{hk} \\ -y^j z^k X^h \left\{ \partial_k \Gamma^i_{jh} - \Gamma^i_{kh} \Gamma^h_{jl} \right\} - t^k X^h \Gamma^i_{kh} \end{pmatrix} \tag{40}$$

with respect to induced coordinates of TTM [4].

Definition 24 Let $\widetilde{X}$ be any vector field on TM. In order to find the restrictions of ${}^2M$ of the differential geometric objects (as function, vector field and one form) on TTM, the coordinates y^i and z^i of these differential geometric objects on TTM are taken as equal. For that reason, the restriction to ${}^2M$ of the horizontal lift of $\widetilde{X}$ to TTM is obtained. As a consequence this fact, $\widetilde{X}^H\big|_{{}^2M}$, the restriction to ${}^2M$ of the vector field $\widetilde{X}^H$ on TTM, is called as the horizontal lift of the vector field $\widetilde{X}$ to ${}^2M$.

Theorem 25 Let X^V, X^C, X^H be vertical, complete and horizontal lift of a vector field of X on M. Then the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of a vector field X on M to ${}^2M$ are

$$i)\ X^{VH}\big|_{{}^2M} = \begin{pmatrix} 0 \\ X^i \\ -y^k X^h \Gamma^i_{kh} \end{pmatrix}, \tag{41}$$

$$ii)\ X^{CH}\big|_{{}^2M} = \begin{pmatrix} X^i \\ y^k\left(\partial_k X^i - X^h \Gamma^i_{kh}\right) \\ -y^j y^k \partial_k\left(X^i \Gamma^h_{ki}\right) - z^k X^h \Gamma^i_{kh} \end{pmatrix}, \tag{42}$$

$$iii)\ X^{HH}\big|_{{}^2M} = \begin{pmatrix} X^i \\ -2y^j X^h \Gamma^i_{jh} \\ -y^j y^k X^h\left\{\partial_k \Gamma^i_{jh} - \Gamma^i_{kh}\Gamma^h_{jr}\right\} - z^k X^h \Gamma^i_{kh} \end{pmatrix}. \tag{43}$$

Proof If we take y^i instead of z^i, $\dfrac{\partial}{\partial y^i}$ instead of $\dfrac{\partial}{\partial z^i}$ and z^i instead of t^i in (38), (39) and (40), the restriction to ${}^2M$ of these vector fields must obtain like the equations given by (41), (42) and (43).

Theorem 26 Let X be a vector field defined on M. As $X^{VH} \neq X^{HV}$ in double tangent bundle, there is the equality $X^{VH}\big|_{{}^2M} = X^{HV}\big|_{{}^2M}$ in ${}^2M$.

Proof Let $X = X^i \dfrac{\partial}{\partial x^i}$ be a vector field on M. The horizontal-vertical lift of X to the TTM is

$$\begin{aligned} X^{HV} = \left(X^H\right)^V &= X^h\left(\frac{\partial}{\partial x^h} - y^k \Gamma^h_{ki}\frac{\partial}{\partial y^h}\right)^V \\ &= X^i \frac{\partial}{\partial z^i} - y^k X^i \Gamma^h_{ki}\frac{\partial}{\partial t^h} \end{aligned} \tag{44}$$

From (38) , we get

$$X^{VH} = X^h \frac{\partial}{\partial y^h} - y^k X^i \Gamma^h_{ki}\frac{\partial}{\partial t^h}. \tag{45}$$

From the equalities (44) and (45) we conclude that $X^{HV} \neq X^{VH}$. But the restrictions of these vector fields to ${}^2M$ are equal.

Theorem 27 Let X be a vector field on M. The complete-horizontal lifts or horizontal-complete lifts of X to TTM or ${}^2M$ is equal.

Proof Taking $X^C : \begin{pmatrix} X^i \\ y^j \nabla_j X^i \end{pmatrix}$ instead of $\widetilde{X}$ in (36), we get

$$X^{HC} : \begin{pmatrix} X^i \\ -y^j X^h \Gamma^i_{jh} \\ z^k \dfrac{\partial X^i}{\partial x^k} \\ -y^j z^k \partial_k \left(X^h \Gamma^i_{jh} \right) - t^k X^h \Gamma^i_{kh} \end{pmatrix}. \tag{46}$$

From (39) and (46), it is seen $X^{HC} \neq X^{CH}$ easily. But the restrictions of these vector fields to ${}^2M$ are equal.

5. Lifted One Forms in Extended Acceleration Bundle

In this section, we will define the horizontal lift of a one form $\widetilde{\omega}$ on TM to ${}^2M$. Then we will obtain the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of a one form ω on M to ${}^2M$.

Definition 28 Let (M, g) be flat Riemann manifold. $\widetilde{\omega}$ a one-form on TM and ∇^S Levi Civita connection with respect to g^S Sasaki-Riemann metric. If $\widetilde{\omega}^H$ provides the following equality:

$$\widetilde{\omega}^H = \widetilde{\omega}^C - \nabla^S_{\widetilde{\gamma}} \widetilde{\omega} \tag{47}$$

$\widetilde{\omega}^H$ is called as the horizontal lift of $\widetilde{\omega}$ to TTM [4].

Theorem 29 The complete lift of the one-form $\widetilde{\omega}$ in TM given by

$$\widetilde{\omega} = \widetilde{\omega}_i dx^i + \widetilde{\omega}_{n+i} \delta y^i,$$

with respect to adapted dual frame of TM has the following matrix representation:

$$\widetilde{\omega}^{C}:\begin{pmatrix} \left\{z^k\partial_k\left(\widetilde{\omega}_i+\widetilde{\omega}_{n+h}y^l\Gamma^h_{jl}\right)+ \right. & \{z^k\partial_k\widetilde{\omega}_{n+i}+ & \widetilde{\omega}_i+\widetilde{\omega}_{n+h}y^l\Gamma^h_{jl} & \widetilde{\omega}_{n+i} \\ \left. +t^k\dot{\partial}_k\left(\widetilde{\omega}_i+\widetilde{\omega}_{n+h}y^l\Gamma^h_{jl}\right)\right\} & +t^k\dot{\partial}_k\widetilde{\omega}_{n+i}\} & & \end{pmatrix}, \quad (48)$$

[4].

Theorem 30 Let (M, g) flat Riemann manifold and ∇^S Levi Civita connection depended on the Sasaki Riemann metric on TM. For any one form $\widetilde{\omega}$ on TM, the one form $\nabla^S_{\widetilde{\gamma}}\widetilde{X}$ on TTM has the following matrix representation

$$\nabla^S_{\widetilde{\gamma}}\widetilde{\omega}=\begin{pmatrix} \left\{z^k\left(\delta_k\widetilde{\omega}_i-\widetilde{\omega}_h\Gamma^h_{kl}\right)+y^lz^k\left(\delta_k\widetilde{\omega}_{n+h}-\widetilde{\omega}_{n+l}\Gamma^l_{kh}\right)\Gamma^h_{jl}+ \right. \\ \left.+\left(t^k+y^lz^k\Gamma^h_{jk}\right)\dot{\partial}_h\widetilde{\omega}_i+\left(t^k+y^lz^k\Gamma^h_{jk}\right)\dot{\partial}_h\widetilde{\omega}_{n+m}y^l\Gamma^m_{lr}\right\} \\ z^k\left(\delta_k\widetilde{\omega}_{n+l}-\widetilde{\omega}_{n+h}\Gamma^h_{kl}\right)+\left(t^k+y^lz^k\Gamma^h_{jk}\right)\dot{\partial}_k\widetilde{\omega}_{n+i} \\ 0 \\ 0 \end{pmatrix}^T \quad (49)$$

where T is used to define transpose of matrix.

Proof Let (M, g) be flat Riemann manifold and g^S the Sasaki Riemann metric on M. ∇^S Levi Civita connection on TM depended on g^S and $\widetilde{\omega}=\widetilde{\omega}_i dx^i+\widetilde{\omega}_{n+l}\delta y^l$ a one form on TM. Then the one form $\nabla^S_{\widetilde{\gamma}}\widetilde{X}$ on TTM is calculated as follow:

$$\begin{aligned}\nabla^S_{\widetilde{\gamma}}\widetilde{\omega}&=\widetilde{\gamma}\left[\nabla_{\delta_k}\left(\widetilde{\omega}_i dx^i+\widetilde{\omega}_{n+l}\delta y^l\right)\otimes dx^k+\nabla_{\dot{\partial}_k}\left(\widetilde{\omega}_i dx^i+\widetilde{\omega}_{n+l}\delta y^l\right)\otimes\delta y^k\right]\\ &=\left\{z^k\left(\delta_k\widetilde{\omega}_i-\widetilde{\omega}_h\Gamma^h_{kl}\right)+y^lz^k\left(\delta_k\widetilde{\omega}_{n+h}-\widetilde{\omega}_{n+l}\Gamma^l_{kh}\right)\Gamma^h_{jl}+\right.\\ &\quad+\left(t^k+y^lz^k\Gamma^h_{jk}\right)\dot{\partial}_h\widetilde{\omega}_i+\left(t^k+y^lz^k\Gamma^h_{jk}\right)\dot{\partial}_h\widetilde{\omega}_{n+m}y^l\Gamma^m_{lr}\Big\}dx^l+\\ &\quad+\left\{z^k\left(\delta_k\widetilde{\omega}_{n+i}-\widetilde{\omega}_{n+h}\Gamma^h_{kl}\right)+\left(t^k+y^lz^k\Gamma^h_{jk}\right)\dot{\partial}_k\widetilde{\omega}_{n+i}\right\}dy^l.\end{aligned}$$

Theorem 31 Let (M, g) be flat Riemann manifold and ∇^S Levi Civita connection depend on the Sasaki Riemann metric g^S on TM. Since the vertical, complete and horizontal lift of the one form ω in M to TM are ω^{V}, ω^{C} and ω^{H}, the horizontal lifts of ω^{V}, ω^{C}, ω^{H}, have the following local matrix representations:

$$i)\ \omega^{VH} = \left(z^k \omega_h \Gamma^h_{kt} \quad 0 \quad \omega_t \quad 0\right), \tag{50}$$

$$ii)\ \omega^{CH} = \left(y^j z^k \partial_k \left(\omega_h \Gamma^h_{jt}\right) + t^k \omega_h \Gamma^h_{kt} \quad z^k \omega_h \Gamma^t_{kh} \quad y^j \partial_j \omega_t \quad \omega_t\right), \tag{51}$$

$$iii)\ \omega^{HH} = \left(y^j z^k \omega_h \left(\partial_k \Gamma^h_{jt} + \Gamma^h_{kl} \Gamma^l_{jt}\right) + t^k \omega_h \Gamma^h_{kt} \quad z^k \omega_h \Gamma^h_{kt} \quad y^j \omega_h \Gamma^h_{jt} \quad \omega_t\right) \tag{52}$$

with respect to induced coordinates of TTM [4].

Definition 32 Let $\widetilde{\omega}$ be any one form on TM. In order to find the restrictions of ${}^2M$ of the differential geometric objects (as function, vector field and one form) on TTM, the coordinates y^i and z^i of these differential geometric objects on TTM are taken as equal. For that reason, the restriction to ${}^2M$ of the horizontal lift of $\widetilde{\omega}$ to TTM is obtained. As a consequence this fact, $\widetilde{\omega}^H\big|_{{}^2M}$, the restriction to ${}^2M$ of the one form $\widetilde{\omega}^H$ on TTM, is called as the horizontal lift of the one form $\widetilde{\omega}$ to ${}^2M$.

Theorem 33 Let $\omega^V, \omega^C, \omega^H$ be vertical, complete and horizontal lift of a one form of $\omega = \omega_h dx^h$ on M. Then the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of the one form ω on M to ${}^2M$ are

$$i)\ \omega^{VH}\big|_{{}^2M} = \left(y^k \omega_h \Gamma^h_{kt} \quad \omega_t \quad 0\right), \tag{53}$$

$$ii)\ \omega^{CH}\big|_{{}^2M} = \left(y^j y^k \partial_k \left(\omega_h \Gamma^h_{jt}\right) + z^k \omega_h \Gamma^h_{kt} \quad y^j \left(\partial_j \omega_t + \omega_h \Gamma^h_{kt}\right) \quad \omega_t\right), \tag{54}$$

$$iii)\ \omega^{HH}\big|_{{}^2M} = \left(y^j y^k \omega_h \left(\partial_k \left(\Gamma^h_{jt}\right) + \Gamma^h_{kl} \Gamma^l_{jt}\right) \quad 2y^j \omega_h \Gamma^h_{jt} \quad \omega_t\right) \tag{55}$$

Proof If we take y^i instead of z^i, dy^i instead of dz^i and z^i instead of t^i in (50), (51) and (52), the restriction to ${}^2M$ of these one forms must obtain like the equations given by (53), (54) and (55).

Theorem 34 Let ω be a one form defined on M. As $\omega^{VH}\big|_{{}^2M} \neq \omega^{HV}\big|_{{}^2M}$ in double tangent bundle, there is the equality $\omega^{VH}\big|_{{}^2M} = \omega^{HV}\big|_{{}^2M}$ in ${}^2M$.

Proof Let $\omega = \omega_h dx^h$ be a one form on M. The horizontal-vertical lift of ω to the TTM is

$$\begin{aligned} \omega^{HV} &= \left(\omega^H\right)^V = \left(y^l \omega_h \Gamma^h_{ik} dx^k + \omega_h dy^h\right)^V \\ &= y^l \omega_h \Gamma^h_{ik} dx^k + \omega_h dy^h. \end{aligned} \tag{56}$$

From (50), we get

$$\omega^{VH} = z^l \omega_k \Gamma^k_{ih} dx^h + \omega_h dz^h \tag{57}$$

From the equalities (56) and (57) we conclude that $\omega^{HV} \neq \omega^{VH}$. But the restriction to ${}^2M$ of these one forms are equal.

Theorem 35 Let ω be a one form on M. As $\omega^{CH} \neq \omega^{HC}$ in double tangent bundle, there is the equality $\omega^{CH}\big|_{{}^2M} = \omega^{HC}\big|_{{}^2M}$ in ${}^2M$.

Proof Taking $\omega^H = \left(y^l \nabla_l \omega_t \quad \omega_t\right)$ instead of $\widetilde{\omega}$ in (48), we get

$$\omega^{HC} = \left(z^k y^l \partial_k\left(\omega_h \Gamma^h_{jt}\right) + t^k \omega_h \Gamma^h_{jt} \quad z^k \partial_k\left(\omega_t\right) \quad y^l \omega_h \Gamma^h_{jt} \quad \omega_t\right) \tag{58}$$

and from (51), we conclude that $\omega^{HC} \neq \omega^{CH}$. But the restriction to ${}^2M$ of these one forms are equal $\omega^{HC}\big|_{{}^2M} = \omega^{CH}\big|_{{}^2M}$.

5. OVERALL CONCLUSIONS

In this paper, we defined the horizontal lifts of a function, a vector field and a one form on TM to ${}^2M$ by using the Sasaki Riemann metric on TM. Then we obtained the vertical-horizontal, complete-horizontal and horizontal-horizontal lifts of these tensor fields on M to ${}^2M$.
By using the same operation, second order lifts of different tensor fields on M to ${}^2M$ can be considered.

6. ACKNOWLEDGEMENTS

The present work was presented at the 12th Mathematics Symposium organized by Hacettepe University.

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ON GENERALIZED JORDAN PREDERIVATIONS AND GENERALIZED PREDERIVATIONS OF JORDAN LIE SUPERALGEBRAS

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Received February 24, 2014

Abstract

In this paper, the concepts of (generalized) (θ, φ)-prederivations and (generalized) Jordan (θ, φ)-prederivations on a Jordan Lie superalgebra are introduced. It is proved that Jordan (θ, φ)-prederivations (resp. generalized Jordan (θ, φ)-prederivations) are (θ, φ)-prederivations (resp. generalized (θ, φ)-prederivations) on a Jordan Lie superalgebra under some conditions. In particular, Jordan θ-prederivations are θ-prederivations on a Jordan Lie superalgebra.

Key words: Derivations, Prederivations, Jordan Lie superalgebras
2010 Mathematics Subject Classification: 17B99

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§1 Introduction

Derivations and generalized derivations are interesting subjects both in mathematics and physics, and there has been a great deal of work concerning them. Leger and Luks investigated the structure of the generalized derivations of Jordan Lie algebras systematically (cf. [11]). Generalized derivations on rings was studied in [12, 16]. Generalized derivations also play a key role in Benoist's study of Levi factors in derivation algebras of nilpotent Lie algebras (cf. [3]). In 1969, Herstein showed that a Jordan derivation of a prime ring of characteristic not 2 must be a derivation(cf. [9]). Brešar, M. together with Vukman, J. generalized Herstein's result to Jordan (Θ, φ)-derivations (cf. [4, 5]). In [1], the authors proved that in a 2-torsion free non-commutative prime ring R, a generalized Jordan (θ, φ)-derivation is a generalized (θ, φ)-derivation when θ is an automorphism of R. We gave some results on generalized derivations of Lie color algebras [8]. Moreover, Jordan θ-derivations and generalized Jordan derivations of Lie triple systems were studied in [15–17].

Prederivations (or Lie triple derivations) of Lie algebras were first introduced by Müller to study bi-invariant semi-Riemannian metrics on Lie groups (cf. [14]). Bajo proved that any real or complex Lie algebras admitting a non-singular prederivation is necessarily nilpotent (cf. [2]). In recent years, there has been an increasing interest in investigating prederivations (cf. [6, 7, 10, 12, 13, 18, 20]).

Jordan Lie superalgebras are the natural generalization of Lie superalgebras, and have important applications both in mathematics and physics. Jordan Lie superalgebras are also interesting from a purely mathematical point of view (cf. [19]). In this paper, we introduce the concepts of (generalized) (θ, φ)-prederivations and (generalized) Jordan (θ, φ)-prederivations

for a Jordan Lie superalgebra and obtain some results concerning Jordan (θ,φ)-prederivations (resp. generalized Jordan (θ,φ)-prederivations) and (θ,φ)-prederivations (resp. generalized (θ,φ)-prederivations) on a Jordan Lie superalgebra.

Throughout this paper, the base field $\mathbf{F}$ is assumed to be of characteristic not equal to 3. We now recall some elementary definitions.

§2 Concepts

Definition 2.1. [19] Let L be a $\mathbf{Z_2}$-graded vector space,i.e., $L = L_0 \oplus L_1$ with grading function $d(x)$ give by$d(x) = 0, x \in L_0; d(x) = 1, x \in L_1$. Suppose that a bilinear product $[.,.] : L \otimes L \to L$ satisfies conditions

(i) $d([x,y]) = \{d(x) + d(y)\} \pmod 2$,

(ii) $[y,x] = -\delta(-1)^{d(x)d(y)}[x,y]$,

(iii) $(-1)^{d(x)d(z)}[[x,y],z]+(-1)^{d(y)d(x)}[[y,z],x]+(-1)^{d(z)d(y)}[[z,x],y] = 0$.

We call L a Jordan Lie superalgebra. Clearly, The case of $\delta = 1$ gives the Lie superalgebra.

Definition 2.2. A prederivation (Lie triple derivation) of a Jordan Lie superalgebra L is a linear mapping $D : L \to L$ such that $D([[x,y],z]) = [[D(x),y],z] + [[x,D(y)],z] + [[x,y],D(z)], \forall x,y,z \in L$.

Definition 2.3. A Jordan prederivation (Jordan triple derivation) of a Jordan Lie superalgebra L is a linear mapping $D' : L \to L$ such that $D'([[x,y],x]) = [[D'(x),y],x] + [[x,D'(y)],x] + [[x,y],D'(x)], \forall x,y,z \in L$.

Definition 2.4. Let L be a Jordan Lie superalgebra and let $D, \theta, \varphi : L \to L$ be homogeneous linear mappings, where $d(\theta) = d(\varphi) = 0$. We denote by $d(D)$ the $\mathbf{Z_2}$-graded degree of a homogeneous linear mapping D of L.

1. D is called a $(\theta,\varphi)_1$**-prederivation** if

$$\begin{aligned} D([[x,y],z]) =& [[D(x),\theta(y)],\varphi(z)] + (-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(z)] \\ &+ (-1)^{d(D)(d(x)+d(y))}[[\theta(x),\varphi(y)],D(z)], \forall x,y,z \in L. \end{aligned}$$

2. D is called a $(\theta,\varphi)_2$-**prederivation** if

$$D([[x,y],z]) = [[D(x),\theta(y)],\theta(z)] + (-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(z)] + (-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\varphi(y)],D(z)], \forall x,y,z \in L.$$

3. D is called a $(\theta,\varphi)_3$-**prederivation** if

$$D([[x,y],z]) = [[D(x),\theta(y)],\theta(z)] + (-1)^{d(D)d(x)}[[\varphi(x),D(y)],\theta(z)] + (-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\varphi(y)],D(z)], \forall x,y,z \in L.$$

In particular, a $(\theta,\varphi)_i$-prederivation D is called a θ-**prederivation** if $\theta = \varphi$, $\forall i = 1,2,3$. It is clear that a $(\theta,\varphi)_i$-prederivation is a prederivation when $\theta = \varphi = 1_T$.

Definition 2.5. Let L be a Jordan Lie superalgebra and let $D,\theta,\varphi : L \to L$ be homogeneous linear mappings, where $d(\theta) = d(\varphi) = 0$.

1. D is called a **Jordan** $(\theta,\varphi)_1$-**prederivation** if

$$D([[x,y],x]) = [[D(x),\theta(y)],\varphi(x)] + (-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(x)] + (-1)^{d(D)(d(x)+d(y))}[[\theta(x),\varphi(y)],D(x)], \forall x,y \in L.$$

2. D is called a **Jordan** $(\theta,\varphi)_2$-**prederivation** if

$$D([[x,y],x]) = [[D(x),\theta(y)],\theta(x)] + (-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(x)] + (-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\varphi(y)],D(x)], \forall x,y \in L.$$

3. D is called a **Jordan** $(\theta,\varphi)_3$-**prederivation** if

$$D([[x,y],x]) = [[D(x),\theta(y)],\theta(x)] + (-1)^{d(D)d(x)}[[\varphi(x),D(y)],\theta(x)] + (-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\varphi(y)],D(x)], \forall x,y \in L.$$

In particular, $\forall i = 1, 2, 3$, a Jordan $(\theta, \varphi)_i$-prederivation D is called a **Jordan θ-prederivation** if $\theta = \varphi$. It is clear that a Jordan $(\theta, \varphi)_i$-prederivation is a Jordan prederivation when $\theta = \varphi = 1_T$.

It is clear that if D_i is a $(\theta, \varphi)_i$-prederivation of L, then D_i is a Jordan $(\theta, \varphi)_i$-prederivation of L, where $i = 1, 2, 3$.

In this paper, L is a Jordan Lie superalgebra and θ, φ are defined to be homogeneous linear mappings of L satisfying $d(\theta) = d(\varphi) = 0$.

Definition 2.6. Let L be a Jordan Lie superalgebra.

1. A **generalized $(\theta, \varphi)_1$-prederivation** with respect to a $(\theta, \varphi)_1$-prederivation E_1 is a homogeneous linear mapping $D_1 : L \to L$ such that $d(D_1) = d(E_1)$ and

$$\begin{aligned} D_1([[x, y], z]) =& [[E_1(x), \theta(y)], \varphi(z)] + (-1)^{d(D_1)d(x)}[[\theta(x), E_1(y)], \varphi(z)] \\ &+ (-1)^{d(D_1)(d(x)+d(y))}[[\theta(x), \varphi(y)], D_1(z)], \forall x, y, z \in L. \end{aligned}$$

2. A **generalized $(\theta, \varphi)_2$-prederivation** with respect to a $(\theta, \varphi)_2$-prederivation E_2 is a homogeneous linear mapping $D_2 : L \to L$ such that $d(D_2) = d(E_2)$ and

$$\begin{aligned} D_2([[x, y], z]) =& [[E_2(x), \theta(y)], \theta(z)] + (-1)^{d(D_2)d(x)}[[\theta(x), E_2(y)], \varphi(z)] \\ &+ (-1)^{d(D_2)(d(x)+d(y))}[[\varphi(x), \varphi(y)], D_2(z)], \forall x, y, z \in L. \end{aligned}$$

3. A **generalized $(\theta, \varphi)_3$-prederivation** with respect to a $(\theta, \varphi)_3$-prederivation δ_3 is a homogeneous linear mapping $D_3 : L \to L$ such that $d(D_3) = d(E_3)$ and

$$\begin{aligned} D_3([[x, y], z]) =& [[E_3(x), \theta(y)], \theta(z)] + (-1)^{d(D_3)d(x)}[[\varphi(x), E_3(y)], \theta(z)] \\ &+ (-1)^{d(D_3)(d(x)+d(y))}[[\varphi(x), \varphi(y)], D_3(z)], \forall x, y, z \in L. \end{aligned}$$

In particular, $\forall i = 1, 2, 3$, a generalized $(\theta, \varphi)_i$-prederivation D is called a **generalized θ-prederivation** with respect to a θ-prederivation E if $\theta = \varphi$.

It is clear that D is a generalized prederivation when $\theta = \varphi = 1_T$ and E is a prederivation.

Definition 2.7. Let L be a Jordan Lie superalgebra.

1. A **generalized Jordan $(\theta, \varphi)_1$-prederivation** with respect to a Jordan $(\theta, \varphi)_1$-prederivation E_1 is a homogeneous linear mapping $D_1 : L \to L$ such that $d(D_1) = d(E_1)$ and

$$\begin{aligned} D_1([[x, y], x]) =& [[E_1(x), \theta(y)], \varphi(x)] + (-1)^{d(D_1)d(x)}[[\theta(x), E_1(y)], \varphi(x)] \\ &+ (-1)^{d(D_1)(d(x)+d(y))}[[\theta(x), \varphi(y)], D_1(x)], \forall x, y \in L. \end{aligned}$$

2. A **generalized Jordan $(\theta, \varphi)_2$-prederivation** with respect to a Jordan $(\theta, \varphi)_2$-prederivation E_2 is a homogeneous linear mapping $D_2 : L \to L$ such that $d(D_2) = d(E_2)$ and

$$\begin{aligned} D_2([[x, y], x]) =& [[E_2(x), \theta(y)], \theta(x)] + (-1)^{d(D_2)d(x)}[[\theta(x), E_2(y)], \varphi(x)] \\ &+ (-1)^{d(D_2)(d(x)+d(y))}[[\varphi(x), \varphi(y)], D_2(x)], \forall x, y \in L. \end{aligned}$$

3. A **generalized Jordan $(\theta, \varphi)_3$-prederivation** with respect to a Jordan $(\theta, \varphi)_3$-prederivation E_3 is a homogeneous linear mapping $D_3 : L \to L$ such that $d(D_3) = d(E_3)$ and

$$\begin{aligned} D_3([[x, y], x]) =& [[E_3(x), \theta(y)], \theta(x)] + (-1)^{d(D_3)d(x)}[[\varphi(x), E_3(y)], \theta(x)] \\ &+ (-1)^{d(D_3)(d(x)+d(y))}[[\varphi(x), \varphi(y)], D_3(x)], \forall x, y \in L. \end{aligned}$$

In particular, $\forall i = 1, 2, 3$, a generalized Jordan $(\theta, \varphi)_i$-prederivation D is called a **generalized Jordan θ-prederivation** with respect to a Jordan θ-prederivation E if $\theta = \varphi$. It is clear that D is a generalized Jordan prederivation when $\theta = \varphi = 1_T$ and E is a Jordan prederivation.

§3 Main Results

Theorem 3.1. D is a $(\theta,\varphi)_1$-prederivation of L if and only if D is a Jordan $(\theta,\varphi)_1$-prederivation of L such that

(i) $[[\theta(x),\varphi(y)],D(z)]=[[\varphi(x),\theta(y)],D(z)]$,

(ii) $(-1)^{d(x)d(z)}A(x,y,z)+(-1)^{d(x)d(y)}A(y,z,x)+(-1)^{d(y)d(z)}A(z,x,y)=0$,

where $x,y,z\in L$ and $A(x,y,z)=[[D(x),\theta(y)],\varphi(z)]+(-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(z)]+(-1)^{d(D)(d(x)+d(y))}[[\theta(x),\varphi(y)],D(z)]$.

Proof. Assume that D is a $(\theta,\varphi)_1$-prederivation of L. Clearly, D is a Jordan $(\theta,\varphi)_1$-prederivation of L and $D([[x,y],z])=-\delta(-1)^{d(x)d(y)}D([[y,x],z])$; note that

$$\begin{aligned}
&-\delta(-1)^{d(x)d(y)}D([[y,x],z])\\
=&-\delta(-1)^{d(x)d(y)}[[D(y),\theta(x)],\varphi(z)]-\delta(-1)^{d(x)d(y)}(-1)^{d(D)d(y)}[[\theta(y),D(x)],\varphi(z)]\\
&-\delta(-1)^{d(x)d(y)}(-1)^{d(D)(d(x)+d(y))}[[\theta(y),\varphi(x)],D(z)]\\
=&[D(x),\theta(y),\varphi(z)]+(-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(z)]\\
&+(-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\theta(y)],D(z)]
\end{aligned}$$

and

$$\begin{aligned}
D([[x,y],z])=&[[D(x),\theta(y)],\varphi(z)]+(-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(z)]\\
&+(-1)^{d(D)(d(x)+d(y))}[[\theta(x),\varphi(y)],D(z)],
\end{aligned}$$

then (i) follows. Since D is a $(\theta,\varphi)_1$-prederivation of L, we have $D([[x,y],z])=A(x,y,z)$; hence

$$\begin{aligned}
&(-1)^{d(x)d(z)}A(x,y,z)+(-1)^{d(x)d(y)}A(y,z,x)+(-1)^{d(y)d(z)}A(z,x,y)\\
=&(-1)^{d(x)d(z)}D([[x,y],z])+(-1)^{d(x)d(y)}D([[y,z],x])+(-1)^{d(y)d(z)}D([[z,x],y])\\
=&D\left((-1)^{d(x)d(z)}[[x,y],z]+(-1)^{d(x)d(y)}[[y,z],x]+(-1)^{d(y)d(z)}[[z,x],y]\right)\\
=&0.
\end{aligned}$$

Conversely, let D be a Jordan $(\theta,\varphi)_1$-prederivation of L for which (i) and (ii) hold. Then $D([[x,y],x]) = A(x,y,x)$. It follows that

$$\begin{aligned}
&D([[x+z,y],x+z]) \\
&= D([[x,y],x]) + D([[x,y],z]) + D([[z,y],x]) + D([[z,y],z]) \\
&= A(x,y,x) + A(z,y,z) + D([[x,y],z]) + D([[z,y],x])
\end{aligned}$$

and

$$\begin{aligned}
D([[x+z,y],x+z]) &= A(x+z,y,x+z) \\
&= A(x,y,x) + A(x,y,z) + A(z,y,x) + A(z,y,z).
\end{aligned}$$

Thus we obtain

$$D([[x,y],z]) + D([[z,y],x]) = A(x,y,z) + A(z,y,x). \qquad (*)$$

By (i),

$$\begin{aligned}
&A(y,x,z) \\
=&[[D(y),\theta(x)],\varphi(z)] + (-1)^{d(D)d(y)}[[\theta(y),D(x)],\varphi(z)] \\
&+ (-1)^{d(D)(d(x)+d(y))}[[\theta(y),\varphi(x)],D(z)] \\
=&-\delta(-1)^{d(x)d(y)}[[D(x),\theta(y)],\varphi(z)] - \delta(-1)^{d(x)d(y)}(-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(z)] \\
&-\delta(-1)^{d(x)d(y)}(-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\theta(y)],D(z)] \\
=&-\delta(-1)^{d(x)d(y)}[[D(x),\theta(y)],\varphi(z)] - \delta(-1)^{d(x)d(y)}(-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(z)] \\
&-\delta(-1)^{d(x)d(y)}(-1)^{d(D)(d(x)+d(y))}[[\theta(x),\varphi(y)],D(z)] \\
=&-\delta(-1)^{d(x)d(y)}A(x,y,z).
\end{aligned}$$

This implies that

$$\begin{aligned}
D([[x,y],y]) &= -\delta(-1)^{d(x)d(y)}D([[y,x],y]) = -\delta(-1)^{d(x)d(y)}A(y,x,y) \\
&= A(x,y,y).
\end{aligned}$$

A similar argument proves

$$D([[x,y],z]) + D([[x,z],y]) = A(x,y,z) + A(x,z,y). \qquad (**)$$

By $(*)\times(-1)^{d(x)d(z)}+(**)\times(-1)^{d(x)d(y)}$, we have

$$\begin{aligned}
&(-1)^{d(x)d(z)}\big(D([[x,y],z]) + D([[z,y],x])\big)\\
&+(-1)^{d(x)d(y)}\big(D([[x,y],z]) + D([[x,z],y])\big)\\
=&(-1)^{d(x)d(z)}\big(A(x,y,z) + A(z,y,x)\big) + (-1)^{d(x)d(y)}\big(A(x,y,z) + A(x,z,y)\big),
\end{aligned}$$

then

$$\begin{aligned}
&(-1)^{d(x)d(z)}D([[x,y],z]) + (-1)^{d(x)d(y)}D([[x,y],z]) - (-1)^{d(y)d(z)}D([[y,x],z])\\
=&(-1)^{d(x)d(z)}\big(A(x,y,z) + A(z,y,x)\big) + (-1)^{d(x)d(y)}\big(A(x,y,z) + A(x,z,y)\big),
\end{aligned}$$

that is,

$$\begin{aligned}
&\Big((-1)^{d(x)d(z)} + (-1)^{d(x)d(y)} + \delta(-1)^{d(y)(d(x)+d(z))}\Big)D([[x,y],z])\\
=&\Big((-1)^{d(x)d(z)} + (-1)^{d(x)d(y)} + \delta(-1)^{d(y)(d(x)+d(z))}\Big)A(x,y,z)\\
&+(-1)^{d(x)d(z)}A(z,y,x) + (-1)^{d(x)d(y)}A(x,z,y) - \delta(-1)^{d(y)(d(x)+d(z))}A(x,y,z)\\
=&\Big((-1)^{d(x)d(z)} + (-1)^{d(x)d(y)} + \delta(-1)^{d(y)(d(x)+d(z))}\Big)A(x,y,z)\\
&+(-1)^{d(x)d(z)}A(z,y,x) + (-1)^{d(x)d(y)}A(x,z,y) + (-1)^{d(y)d(z)}A(y,x,z)\\
=&\Big((-1)^{d(x)d(z)} + (-1)^{d(x)d(y)} + \delta(-1)^{d(y)(d(x)+d(z))}\Big)A(x,y,z),
\end{aligned}$$

where the last equality uses (ii). Since ch$\mathbf{F} \neq 3$, we have $(-1)^{d(x)d(z)} + (-1)^{d(x)d(y)} + \delta(-1)^{d(y)(d(x)+d(z))} \neq 0$, and so $D([[x,y],z]) = A(x,y,z)$, i.e., D is a $(\theta,\varphi)_1$-prederivation of L.

Corollary 3.2. D is a θ-prederivation of L if and only if D is a Jordan θ-prederivation of L.

Proof. If D is a Jordan θ-prederivation of L, then (i) follows immediately. (ii) holds because

$$
\begin{aligned}
&(-1)^{d(x)d(z)}A(x,y,z)+(-1)^{d(x)d(y)}A(y,z,x)+(-1)^{d(y)d(z)}A(z,x,y)\\
=&(-1)^{d(x)d(z)}[[D(x),\theta(y)],\theta(z)]+(-1)^{d(x)d(z)}(-1)^{d(D)d(x)}[[\theta(x),D(y)],\theta(z)]\\
&+(-1)^{d(x)d(z)}(-1)^{d(D)(d(x)+d(y))}[[\theta(x),\theta(y)],D(z)]\\
&+(-1)^{d(x)d(y)}[[D(y),\theta(z)],\theta(x)]+(-1)^{d(x)d(y)}(-1)^{d(D)d(y)}[[\theta(y),D(z)],\theta(x)]\\
&+(-1)^{d(x)d(y)}(-1)^{d(D)(d(y)+d(z))}[[\theta(y),\theta(z)],D(x)]\\
&+(-1)^{d(y)d(z)}[[D(z),\theta(x)],\theta(y)]+(-1)^{d(y)d(z)}(-1)^{d(D)d(z)}[[\theta(z),D(x)],\theta(y)]\\
&+(-1)^{d(y)d(z)}(-1)^{d(D)(d(x)+d(z))}[[\theta(z),\theta(x)],D(y)]\\
=&(-1)^{d(D)d(z)}\big((-1)^{d(z)(d(D)+d(x))}[[D(x),\theta(y)],\theta(z)]\\
&+(-1)^{d(y)(d(D)+d(x))}[[\theta(y),\theta(z)],D(x)]+(-1)^{d(y)d(z)}[[\theta(z),D(x)],\theta(y)]\big)\\
&+(-1)^{d(D)d(x)}\big((-1)^{d(x)(d(D)+d(y))}[[D(y),\theta(z)],\theta(x)]\\
&+(-1)^{d(z)(d(D)+d(y))}[[\theta(z),\theta(x)],D(y)]+(-1)^{d(x)d(z)}[[\theta(x),D(y)],\theta(z)]\big)\\
&+(-1)^{d(D)d(y)}\big((-1)^{d(y)(d(D)+d(z))}[[D(z),\theta(x)],\theta(y)]\\
&+(-1)^{d(x)(d(D)+d(z))}[[\theta(x),\theta(y)],D(z)]+(-1)^{d(x)d(y)}[[\theta(y),D(z)],\theta(x)]\big)\\
=&0.
\end{aligned}
$$

Therefore, D is a θ-prederivation of L by Theorem 3.1.

Theorem 3.3. D is a $(\theta,\varphi)_2$-prederivation of L if and only if D is a Jordan $(\theta,\varphi)_2$-prederivation of L such that

(i) $[[D(x),\theta(y)],(\varphi-\theta)(z)]=(-1)^{d(D)d(x)}[[\theta(x),D(y)],(\varphi-\theta)(z)]$,

(ii) $(-1)^{d(x)d(z)}A'(x,y,z)+(-1)^{d(x)d(y)}A'(y,z,x)+(-1)^{d(y)d(z)}A'(z,x,y)=0$,

where $x,y,z\in L$ and $A'(x,y,z)=[[D(x),\theta(y)],\theta(z)]+(-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(z)]+(-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\varphi(y)],D(z)]$.

Proof. Let D be a $(\theta,\varphi)_2$-prederivation of L. Use the fact that

$$D([[x,y],z]) = -\delta(-1)^{d(x)d(y)}D([[y,x],z]),$$

as well as the fact that

$$\begin{aligned}
&-\delta(-1)^{d(x)d(y)}D([[y,x],z])\\
=&-\delta(-1)^{d(x)d(y)}[[D(y),\theta(x)],\theta(z)]-\delta(-1)^{d(x)d(y)}(-1)^{d(D)d(y)}[[\theta(y),D(x)],\varphi(z)]\\
&-\delta(-1)^{d(x)d(y)}(-1)^{d(D)(d(x)+d(y))}[[\varphi(y),\varphi(x)],D(z)]\\
=&[[D(x),\theta(y)],\varphi(z)]+(-1)^{d(D)d(x)}[[\theta(x),D(y)],\theta(z)]\\
&+(-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\varphi(y)],D(z)],
\end{aligned}$$

then we have

$$\begin{aligned}
&[[D(x),\theta(y)],\theta(z)]+(-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(z)]\\
=&[[D(x),\theta(y)],\varphi(z)]+(-1)^{d(D)d(x)}[[\theta(x),D(y)],\theta(z)],
\end{aligned}$$

that is, $[[D(x),\theta(y)],(\varphi-\theta)(z)] = (-1)^{d(D)d(x)}[[\theta(x),D(y)],(\varphi-\theta)(z)]$. It is routine to prove (ii).

Suppose, conversely, that D is a Jordan $(\theta,\varphi)_2$-prederivation of L satisfying (i) and (ii). Note that

$$\begin{aligned}
&A'(y,x,z)\\
=&[[D(y),\theta(x)],\theta(z)]+(-1)^{d(D)d(y)}[[\theta(y),D(x)],\varphi(z)]\\
&+(-1)^{d(D)(d(x)+d(y))}[[\varphi(y),\varphi(x)],D(z)]\\
=&-\delta(-1)^{d(x)d(y)}[[D(x),\theta(y)],\varphi(z)]-\delta(-1)^{d(x)d(y)}(-1)^{d(D)d(x)}[[\theta(x),D(y)],\theta(z)]\\
&-\delta(-1)^{d(x)d(y)}(-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\varphi(y)],D(z)]\\
=&-\delta(-1)^{d(x)d(y)}[[D(x),\theta(y)],\theta(z)]-\delta(-1)^{d(x)d(y)}(-1)^{d(D)d(x)}[[\theta(x),D(y)],\varphi(z)]\\
&-\delta(-1)^{d(x)d(y)}(-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\varphi(y)],D(z)]\\
=&-\delta(-1)^{d(x)d(y)}A'(x,y,z).
\end{aligned}$$

In the same way, we can get equalities (*) and (**). The rest proof is the same as the corresponding proof of Theorem 3.1.

A similar argument proves the following result.

Theorem 3.4. D is a $(\theta,\varphi)_3$-prederivation of L if and only if D is a Jordan $(\theta,\varphi)_3$-prederivation of L such that

(i) $[[D(x),(\theta-\varphi)(y)],\theta(z)] = (-1)^{d(D)d(x)}[[(\theta-\varphi)(x),D(y)],\theta(z)]$,

(ii) $(-1)^{d(x)d(z)}A''(x,y,z)+(-1)^{d(x)d(y)}A''(y,z,x)+(-1)^{d(y)d(z)}A''(z,x,y) = 0$,

where $x,y,z \in L$ and $A''(x,y,z)=[[D(x),\theta(y)],\theta(z)]+(-1)^{d(D)d(x)}[[\varphi(x),D(y)],\theta(z)] + (-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\varphi(y)],D(z)]$.

Remark 3.5. Corollary 3.2 can also be concluded from Theorem 3.3 or Theorem 3.4 since for any $x,y,z \in L$, $A(x,y,z) = A'(x,y,z) = A''(x,y,z)$ when D is a Jordan θ-prederivation.

Theorem 3.6. D is a generalized $(\theta,\varphi)_1$-prederivation of L with respect to a $(\theta,\varphi)_1$-prederivation E if and only if D is a generalized Jordan $(\theta,\varphi)_1$-prederivation of L with respect to a $(\theta,\varphi)_1$-prederivation E such that

(i) $[[\theta(x),\varphi(y)],D(z)] = [[\varphi(x),\theta(y)],D(z)]$,

(ii) $(-1)^{d(x)d(z)}B(x,y,z)+(-1)^{d(x)d(y)}B(y,z,x)+(-1)^{d(y)d(z)}B(z,x,y) = 0$,

where $x,y,z \in L$ and $B(x,y,z) = [[E(x),\theta(y)],\varphi(z)]+(-1)^{d(D)d(x)}[[\theta(x),E(y)],\varphi(z)] + (-1)^{d(D)(d(x)+d(y))}[[\theta(x),\varphi(y)],D(z)]$.

Proof. Suppose that D is a generalized $(\theta,\varphi)_1$-prederivation of L with respect to a $(\theta,\varphi)_1$-prederivation E. Clearly, D is a generalized Jordan $(\theta,\varphi)_1$-prederivation of L and $D([x,y,z]) = -\delta(-1)^{d(x)d(y)}D([y,x,z])$. (i)

follows from the fact

$$\begin{aligned}
&-\delta(-1)^{d(x)d(y)}D([[y,x],z])\\
=&-\delta(-1)^{d(x)d(y)}[[E(y),\theta(x)],\varphi(z)]-\delta(-1)^{d(x)d(y)}(-1)^{d(D)d(y)}[[\theta(y),E(x)],\varphi(z)]\\
&-\delta(-1)^{d(x)d(y)}(-1)^{d(D)(d(x)+d(y))}[[\theta(y),\varphi(x)],D(z)]\\
=&[[E(x),\theta(y)],\varphi(z)]+(-1)^{d(D)d(x)}[[\theta(x),E(y)],\varphi(z)]\\
&+(-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\theta(y)],D(z)].
\end{aligned}$$

Since D is a generalized $(\theta,\varphi)_1$-prederivation of L with respect to E, we have $D([[x,y],z])=B(x,y,z)$; hence

$$\begin{aligned}
&(-1)^{d(x)d(z)}B(x,y,z)+(-1)^{d(x)d(y)}B(y,z,x)+(-1)^{d(y)d(z)}B(z,x,y)\\
=&(-1)^{d(x)d(z)}D([[x,y],z])+(-1)^{d(x)d(y)}D([[y,z],x])+(-1)^{d(y)d(z)}D([[z,x],y])\\
=&D\left((-1)^{d(x)d(z)}[[x,y],z]+(-1)^{d(x)d(y)}[[y,z],x]+(-1)^{d(y)d(z)}[[z,x],y]\right)\\
=&0.
\end{aligned}$$

Conversely, if D is a generalized Jordan $(\theta,\varphi)_1$-prederivation of L with respect to a $(\theta,\varphi)_1$-prederivation E satisfying (i) and (ii), then, refer to the proof of Theorem 3.1, it suffices to prove $B(x,y,z)=-\delta(-1)^{d(x)d(y)}B(y,x,z)$. In fact,

$$\begin{aligned}
&B(y,x,z)\\
=&[[E(y),\theta(x)],\varphi(z)]+(-1)^{d(D)d(y)}[[\theta(y),E(x)],\varphi(z)]\\
&+(-1)^{d(D)(d(x)+d(y))}[[\theta(y),\varphi(x)],D(z)]\\
=&-\delta(-1)^{d(x)d(y)}[[E(x),\theta(y)],\varphi(z)]-\delta(-1)^{d(x)d(y)}(-1)^{d(D)d(x)}[[\theta(x),E(y)],\varphi(z)]\\
&-\delta(-1)^{d(x)d(y)}(-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\theta(y)],D(z)]\\
=&-\delta(-1)^{d(x)d(y)}[[E(x),\theta(y)],\varphi(z)]-\delta(-1)^{d(x)d(y)}(-1)^{d(D)d(x)}[[\theta(x),E(y)],\varphi(z)]\\
&-\delta(-1)^{d(x)d(y)}(-1)^{d(D)(d(x)+d(y))}[[\theta(x),\varphi(y)],D(z)]\\
=&-\delta(-1)^{d(x)d(y)}B(x,y,z).
\end{aligned}$$

This completes the proof.

Corollary 3.7. D is a generalized θ-prederivation of L with respect to a θ-prederivation E if and only if D is a generalized Jordan θ-prederivation of L with respect to a Jordan θ-prederivation E such that

$$\begin{aligned}&(-1)^{d(z)(d(D)+d(x))}[[\theta(x),\theta(y)],(D-E)(z)]\\+&(-1)^{d(x)(d(D)+d(y))}[[\theta(y),\theta(z)],(D-E)(x)]\\+&(-1)^{d(y)(d(D)+d(z))}[[\theta(z),\theta(x)],(D-E)(y)]=0.\end{aligned}$$

Proof. If D is a generalized Jordan θ-prederivation of L, then (i) follows immediately. (ii) holds because

$$\begin{aligned}&(-1)^{d(x)d(z)}B(x,y,z)+(-1)^{d(x)d(y)}B(y,z,x)+(-1)^{d(y)d(z)}B(z,x,y)\\=&(-1)^{d(x)d(z)}[[E(x),\theta(y)],\theta(z)]+(-1)^{d(x)d(z)}(-1)^{d(D)d(x)}[[\theta(x),E(y)],\theta(z)]\\&+(-1)^{d(x)d(z)}(-1)^{d(D)(d(x)+d(y))}[[\theta(x),\theta(y)],D(z)]\\&+(-1)^{d(x)d(y)}[[E(y),\theta(z)],\theta(x)]+(-1)^{d(x)d(y)}(-1)^{d(D)d(y)}[[\theta(y),E(z)],\theta(x)]\\&+(-1)^{d(x)d(y)}(-1)^{d(D)(d(y)+d(z))}[[\theta(y),\theta(z)],D(x)]\\&+(-1)^{d(y)d(z)}[[E(z),\theta(x)],\theta(y)]+(-1)^{d(y)d(z)}(-1)^{d(D)d(z)}[[\theta(z),E(x)],\theta(y)]\\&+(-1)^{d(y)d(z)}(-1)^{d(D)(d(x)+d(z))}[[\theta(z),\theta(x)],D(y)]\\=&(-1)^{d(x)d(z)}[[E(x),\theta(y)],\theta(z)]+(-1)^{d(x)d(z)}(-1)^{d(D)d(x)}[[\theta(x),E(y)],\theta(z)]\\&+(-1)^{d(x)d(z)}(-1)^{d(D)(d(x)+d(y))}[[\theta(x),\theta(y)],E(z)]\\&+(-1)^{d(x)d(y)}[[E(y),\theta(z)],\theta(x)]+(-1)^{d(x)d(y)}(-1)^{d(D)d(y)}[[\theta(y),E(z)],\theta(x)]\\&+(-1)^{d(x)d(y)}(-1)^{d(D)(d(y)+d(z))}[[\theta(y),\theta(z)],E(x)]\\&+(-1)^{d(y)d(z)}[[E(z),\theta(x)],\theta(y)]+(-1)^{d(y)d(z)}(-1)^{d(D)d(z)}[[\theta(z),E(x)],\theta(y)]\\&+(-1)^{d(y)d(z)}(-1)^{d(D)(d(x)+d(z))}[[\theta(z),\theta(x)],E(y)]=\end{aligned}$$

$$
\begin{aligned}
=&(-1)^{d(D)d(z)}\left((-1)^{d(z)(d(D)+d(x))}[[E(x),\theta(y)],\theta(z)]\right.\\
&+(-1)^{d(y)(d(D)+d(x))}[[\theta(y),\theta(z)],E(x)]+(-1)^{d(y)d(z)}[[\theta(z),E(x)],\theta(y)]\big)\\
&+(-1)^{d(D)d(x)}\left((-1)^{d(x)(d(D)+d(y))}[[E(y),\theta(z)],\theta(x)]\right.\\
&+(-1)^{d(z)(d(D)+d(y))}[[\theta(z),\theta(x)],E(y)]+(-1)^{d(x)d(z)}[[\theta(x),E(y)],\theta(z)]\big)\\
&+(-1)^{d(D)d(y)}\left((-1)^{d(y)(d(D)+d(z))}[[E(z),\theta(x)],\theta(y)]\right.\\
&+(-1)^{d(x)(d(D)+d(z))}[[\theta(x),\theta(y)],E(z)]+(-1)^{d(x)d(y)}[[\theta(y),E(z)],\theta(x)]\big)\\
=&0.
\end{aligned}
$$

Note that E is a θ-prederivation by Corollary 3.2. Therefore, D is a generalized θ-prederivation of L with respect to a θ-prederivation E by Theorem 3.6.

As before, one can prove the following theorems.

Theorem 3.8. D is a generalized $(\theta,\varphi)_2$-prederivation of L with respect to a $(\theta,\varphi)_2$-prederivation E if and only if D is a generalized Jordan $(\theta,\varphi)_2$-prederivation of L with respect to a $(\theta,\varphi)_2$-prederivation E such that

(i) $[[E(x),\theta(y)],(\varphi-\theta)(z)]=(-1)^{d(D)d(x)}[[\theta(x),E(y)],(\varphi-\theta)(z)]$,

(ii) $(-1)^{d(x)d(z)}B'(x,y,z)+(-1)^{d(x)d(y)}B'(y,z,x)+(-1)^{d(y)d(z)}B'(z,x,y)=0$,

where $x,y,z\in L$ and $B'(x,y,z)=[[E(x),\theta(y)],\theta(z)]+(-1)^{d(D)d(x)}[[\theta(x),E(y)],\varphi(z)]+(-1)^{d(D)(d(x)+d(y))}[[\varphi(x),\varphi(y)],D(z)]$.

Theorem 3.9. D is a generalized $(\theta,\varphi)_3$-prederivation of L with respect to a $(\theta,\varphi)_3$-prederivation E if and only if D is a generalized Jordan $(\theta,\varphi)_3$-prederivation of L with respect to a $(\theta,\varphi)_3$-prederivation E such that

(i) $[[E(x),(\theta-\varphi)(y)],\theta(z)]=(-1)^{d(D)d(x)}[[(\theta-\varphi)(x),E(y)],\theta(z)]$,

(ii) $(-1)^{d(x)d(z)}B''(x,y,z)+(-1)^{d(x)d(y)}B''(y,z,x)+(-1)^{d(y)d(z)}B''(z,x,y)=0$,

where $x, y, z \in L$ and $B''(x,y,z) = [[E(x), \theta(y)], \theta(z)] + (-1)^{d(E)d(x)}[[\varphi(x), E(y)], \theta(z)] + (-1)^{d(D)(d(x)+d(y))}[[\varphi(x), \varphi(y)], D(z)]$.

Remark 3.10. Corollary 3.7 can also be concluded from Theorem 3.8 or Theorem 3.9, since for any $x, y, z \in L$, $B(x,y,z) = B'(x,y,z) = B''(x,y,z)$ when D is a generalized Jordan θ-prederivation.

Acknowledgements This work was supported by the National Natural Science Foundation of China (No.11171055), the Natural Science Foundation of Jilin province (No.201115006), Scientific Research Foundation for Returned Scholars Ministry of Education of China and the Fundamental Research Funds for the Central Universities (No.12SSXT139).

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ALGEBRAS GROUPS AND GEOMETRIES 31 63 - 100 (2014)

ON A NEW DIFFERENTIAL GEOMETRIC METHOD IN METRIC-TORSION THEORIES OF GRAVITATION

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Received February 24, 2014

Abstract

In this paper a new differential geometric method in a form of the manifestly generally covariant formalism for constructing conservation laws and conserved quantities in arbitrary metric-torsion theories of gravitation is presented.

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I. INTRODUCTION

Last decades, one can see an unprecedented active development of alternative theories of gravity, which modify general relativity (GR) in various ways. Among them there are scalar-tensor theories, the Einstein-Cartan theory, the Lovelock theory in the general form as well as its special cases, such as very popular Einstein-Gauss-Bonnet gravity, metric-affine theories, supergravity, $f(R)$-theories, Chern-Simons modifications of GR, Lovelock-Cartan theories, topologically massive gravity, topologically massive supergravity, new massive gravity, critical gravity, chiral gravity, various topological gauge theories of gravity and supergravity, etc.

Constructing the conservation laws (CLs) and conserved quantities (CQs) in an arbitrary field theory, including gravitational theories, is a main problem. Many above listed theories, presented in the second order formalism, are the metric-torsion theories. Therefore, there is a demand in *universal* expressions for CLs and CQs. Recently such a formalism was developed in papers of the authors [1–3]. The formalism itself is an initially manifestly covariant modification of the Noether-like approach suggested by Bergmann [4, 5] and Mitskievich [6–8]. In the present works, we give a brief derivation of this formalism and new results obtained in its framework. The novelty of our results is in the following:

- *Universality.* We consider an *arbitrary* diffeomorphically invariant classical field theories, Lagrangians of which contain derivatives of field variables (tensor densities of *arbitrary*, but fixed ranks and weights) up to the second order;

- *Manifest general covariance.* We develop manifestly generally covariant formalism, first, using *initially* generally covariant expressions (without using auxiliary structures, such as a background metric); second, all of our calculations are manifestly generally covariant at *each and every steps*;

- *The torsion field is taken into account.* A spacetime under consideration is presented by an *arbitrary Riemann-Cartan space.* Both the torsion tensor and the metric tensor are the dynamical fields, the torsion coupling in the Lagrangian can be both minimal (through connection) and non-minimal (explicit).

In the paper we use the following notations: Greek indexes α, β, ..., μ, ν, ...take values of 0, 1, ..., D and numerate spacetime coordinates $x \stackrel{def}{=} \{x^{\alpha}\}$, partial $\boldsymbol{\partial} \stackrel{def}{=} \{\partial_{\alpha}\} \stackrel{def}{=} \{\partial/\partial x^{\alpha}\}$ and covariant $\boldsymbol{\nabla} \stackrel{def}{=} \{\nabla_{\alpha}\}$, $\overset{*}{\boldsymbol{\nabla}} \stackrel{def}{=} \{\overset{*}{\nabla}_{\alpha}\}$ derivatives, and spacetime tensor components of fields also. Coordinate x^0 is a time one, whereas coordinates $\vec{x} \stackrel{def}{=} \{x^1, \ldots, x^D\}$ are space ones. Capital Latin indexes A, B, ..., are collective and numerate components of the full set of the physical fields $\boldsymbol{\Phi} \stackrel{def}{=} \{\Phi^A(x)\}$ (containing both gravitational and matter fields) and are related to 1, 2, ..., N. At last, small Latin indexes from the beginning of the alphabet a, b, ..., h numerate components of matter (non-gravitational) fields $\boldsymbol{\varphi} \stackrel{def}{=} \{\varphi^a(x)\}$ and take values of 1, 2, ..., n.

As usual, for a twice repeated index, the Einstein summation rule is assumed. Indexes in parentheses need to be symmetrized; whereas, indexes in brackets needs to be antisymmetrized. Two vertical lines inside the

brackets () and [] mean that indexes between them do not participate in symmetrization/antisymmetrization. For example

$$A_{(\alpha\beta)} = \tfrac{1}{2}\left(A_{\alpha\beta} + A_{\beta\alpha}\right), A_{[\alpha|\beta|\gamma]} = \tfrac{1}{2}\left(A_{\alpha\beta\gamma} - A_{\gamma\beta\alpha}\right).$$

The speed of light in vacuum is set to one.

Important definitions and relations in the Riemann-Cartan geometry are given in Ref. [1]. Now, we introduce necessary notations only. The torsion tensor $\mathbf{T} \stackrel{def}{=} \{T^{\lambda}{}_{\mu\nu}\}$ and the curvature tensor $\mathbf{R} \stackrel{def}{=} \{R^{\kappa}{}_{\lambda\mu\nu}\}$ are presented as

$$T^{\lambda}{}_{\mu\nu} = -2\Gamma^{\lambda}{}_{[\mu\nu]};$$

$$R^{\kappa}{}_{\lambda\mu\nu} = 2\left(\partial_{[\mu}\Gamma^{\kappa}{}_{\lambda\nu]} + \Gamma^{\kappa}{}_{\alpha[\mu}\Gamma^{\alpha}{}_{\lambda\nu]}\right).$$

Here, the connection $\mathbf{\Gamma} \stackrel{def}{=} \{\Gamma^{\lambda}{}_{\mu\nu}\}$ is defined by a metric compatible condition

$$\nabla_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\lambda} g_{\alpha\nu} - \Gamma^{\alpha}{}_{\nu\lambda} g_{\mu\alpha} = 0,$$

where the standard covariant derivative $\mathbf{\nabla} \stackrel{def}{=} \{\nabla_{\lambda}\}$ is used and $\mathbf{g} \stackrel{def}{=} \{g_{\mu\nu}\}$ is the metric tensor. The modified covariant derivative $\overset{*}{\mathbf{\nabla}} \stackrel{def}{=} \{\overset{*}{\nabla}_{\lambda}\}$ is

$$\overset{*}{\nabla}_{\lambda} \stackrel{def}{=} \nabla_{\lambda} + T_{\lambda}; \qquad T_{\lambda} \stackrel{def}{=} T^{\alpha}{}_{\lambda\alpha}.$$

II. GENERAL CONSIDERATION

A. The General Noether Identity. Generalized Noether's Current and Charge

We consider a classical field theory determined by the action functional

$$I[\mathbf{\Phi};\Sigma_{1,2}] = \int\limits_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\mathscr{L}, \tag{1}$$

in a Riemann-Cartan space $\mathcal{C}(1,D)$. Here, $dx \stackrel{def}{=} dx^0 dx^1 \ldots dx^D$; integration is provided over an arbitrary $(D+1)$-dimensional volume in $\mathcal{C}(1,D)$ restricted by two spacelike D-dimensional hypersurfaces Σ_1 and Σ_2; Lagrangian $\mathscr{L}$ is a local function of a set of field variables $\mathbf{\Phi}(x) = \{\Phi^A(x); A = \overline{1,N}\}$ and their first and second derivatives. We assume that the Lagrangian $\mathscr{L}$ is a generally covariant scalar and functional variation $\delta_\Phi I$ of the action functional I,

$$\begin{aligned}\delta_\Phi I[\mathbf{\Phi};\Sigma_{1,2}] &\stackrel{def}{=} I[\mathbf{\Phi}+\delta\mathbf{\Phi};\Sigma_{1,2}] - I[\mathbf{\Phi};\Sigma_{1,2}] \\ &= \int\limits_{\Sigma_1}^{\Sigma_2} dx\,\delta\left(\sqrt{-g}\mathscr{L}\right),\end{aligned} \tag{2}$$

has the following structure:

$$\begin{aligned}\delta_\Phi I &= \int\limits_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\frac{\Delta I}{\Delta\Phi^A}\delta\Phi^A \\ &+ \int\limits_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\overset{*}{\nabla}_\mu\left\{K^\mu|_A\delta\Phi^A + L^{\beta\mu}|_A\nabla_\beta\delta\Phi^A\right\}.\end{aligned} \tag{3}$$

Hereinafter, $\Delta I/\Delta\Phi^A$ is defined by the variational derivative $\delta I/\delta\Phi^A$, which is the operator of equations of motion,

$$\frac{\Delta I}{\Delta\Phi^A} \stackrel{def}{=} \frac{1}{\sqrt{-g}}\frac{\delta I}{\delta\Phi^A}, \tag{4}$$

$\mathbf{K} \stackrel{def}{=} \{K^{\mu}|_{A}\}$ and $\mathbf{L} \stackrel{def}{=} \{L^{\beta\mu}|_{A}\}$ are local functions of the field variables $\mathbf{\Phi}$ and their first and second derivatives, and are defined in an unique way (without ambiguities) by the Lagrangian $\mathscr{L}$.

Then, due to the invariance of the action functional I (1) with respect to general/arbitrary infinitesimal diffeomorphisms

$$\begin{cases} \delta x^{\mu} = \delta\xi^{\mu}(x); & (5) \\ \delta_{\xi}\Phi^{A}(x) = \Phi_{\alpha}|^{A}\delta\xi^{\alpha} + \Phi_{\alpha}{}^{\beta}|^{A}\nabla_{\beta}\delta\xi^{\alpha}, & (6) \end{cases}$$

the *general Noether identity* (the main identity) follows:

$$\overset{*}{\nabla}_{\mu} J^{\mu}[\delta\boldsymbol{\xi}] + \frac{\Delta I}{\Delta\Phi^{A}}\delta_{\xi}\Phi^{A} \equiv 0. \qquad (7)$$

In the formula (6) the quantities $\{\Phi_{\alpha}|^{A}\}$, $\{\Phi_{\alpha}{}^{\beta}|^{A}\}$ are local functions of $\mathbf{\Phi}$ and their derivatives, which are defined uniquely by the transformation properties of $\mathbf{\Phi}$, and

$$\begin{aligned} & J^{\mu}[\delta\boldsymbol{\xi}] \\ & \stackrel{def}{=} \left\{K^{\mu}|_{A}\Phi_{\alpha}|^{A} + \mathscr{L}\delta^{\mu}_{\alpha} + L^{\nu\mu}|_{A}\nabla_{\nu}\Phi_{\alpha}|^{A}\right\}\delta\xi^{\alpha} \\ & + \left\{K^{\mu}|_{A}\Phi_{\alpha}{}^{\beta}|^{A} + L^{\beta\mu}|_{A}\Phi_{\alpha}|^{A} + L^{\nu\mu}|_{A}\nabla_{\nu}\Phi_{\alpha}{}^{\beta}|^{A}\right\} \\ & \times\nabla_{\beta}\delta\xi^{\alpha} + \left\{L^{\gamma\mu}|_{A}\Phi_{\alpha}{}^{\beta}|^{A}\right\}\nabla_{\gamma}\nabla_{\beta}\delta\xi^{\alpha}. \end{aligned} \qquad (8)$$

Unlike the standard Noether currents obtained by the recipe of the 1-st Noether theorem in the theories with the global symmetries, the current $\mathbf{J}[\delta\boldsymbol{\xi}] \stackrel{def}{=} \{J^{\mu}[\delta\boldsymbol{\xi}]\}$ (8) depends on an arbitrary *displacement vector* field $\delta\boldsymbol{\xi} \stackrel{def}{=} \{\delta\xi^{\mu}\}$ and its covariant derivatives. For this reason we will call $\mathbf{J}[\delta\boldsymbol{\xi}]$ as the *generalized Noether current*, and a correspondent conserved charge

$$Q[\delta\boldsymbol{\xi};\Sigma] \stackrel{def}{=} \int\limits_{\Sigma} d\sigma_{\mu}\, J^{\mu}[\delta\boldsymbol{\xi}] \qquad (9)$$

as the *generalized Noether charge.*

If equations of motion $\Delta I/\Delta\Phi^A = 0$ hold then the identity (7) transforms into the continuity equation

$$\overset{*}{\nabla}_\mu J^\mu[\delta\boldsymbol{\xi}] = 0 \quad \text{(on the } \boldsymbol{\Phi}\text{-equations)}. \tag{10}$$

If additionally the field variables and their derivatives vanish fast enough at a spatial infinity, then last equation leads to conservation of the generalized charge

$$\delta Q[\delta\boldsymbol{\xi};\, \Sigma]/\delta\Sigma(x) = 0 \quad \text{(on the } \boldsymbol{\Phi}\text{-equations)} \tag{11}$$

meaning that its value is the same on each of hypersurfaces Σ.

B. The Klein and Noether Identities

After some transformations the generalized Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (8) can be led to the next *standard* form:

$$J^\mu[\delta\boldsymbol{\xi}] = U_\alpha{}^\mu\delta\xi^\alpha + M_\alpha{}^{\beta\mu}\nabla_\beta\delta\xi^\alpha + N_\alpha{}^{\beta\gamma\mu}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha \tag{12}$$

where

$$\begin{cases} U_\alpha{}^\mu \overset{def}{=} \mathscr{L}\delta^\mu_\alpha + K^\mu|_A\Phi_\alpha|^A \\ \qquad + L^{\kappa\mu}|_A\left(\nabla_\kappa\Phi_\alpha|^A + \dfrac{1}{2}R^\varepsilon{}_{\alpha\kappa\lambda}\Phi_\varepsilon{}^\lambda|^A\right); & (13) \\ M_\alpha{}^{\beta\mu} \overset{def}{=} K^\mu|_A\Phi_\alpha{}^\beta|^A + L^{\beta\mu}|_A\Phi_\alpha|^A \\ \qquad + L^{\kappa\mu}|_A\left(\nabla_\kappa\Phi_\alpha{}^\beta|^A - \dfrac{1}{2}T^\beta{}_{\kappa\lambda}\Phi_\alpha{}^\lambda|^A\right); & (14) \\ N_\alpha{}^{\beta\gamma\mu} \overset{def}{=} L^{(\gamma|\mu}|_A\Phi_\alpha{}^{|\beta)}|^A. & (15) \end{cases}$$

Substituting (12) and (6) into (7) and taking into account that $\{\delta\xi^\alpha, \nabla_\beta\delta\xi^\alpha, \nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha, \nabla_{(\delta}\nabla_\gamma\nabla_{\beta)}\delta\xi^\alpha\}$ presents a set of independent and arbitrary quantities in every point, one obtains the system of identities:

$$\begin{cases} \overset{*}{\nabla}_\mu U_\alpha{}^\mu - \frac{1}{2}M_\lambda{}^{\mu\nu}R^\lambda{}_{\alpha\mu\nu} - \frac{1}{3}N_\kappa{}^{\lambda\mu\nu}\left(\nabla_\lambda R^\kappa{}_{\alpha\mu\nu} + \frac{1}{2}T^\sigma{}_{\mu\nu}R^\kappa{}_{\alpha\lambda\sigma}\right) \equiv -I_\alpha; \\ U_\alpha{}^\beta + \left(\overset{*}{\nabla}_\mu M_\alpha{}^{\beta\mu} + \frac{1}{2}M_\alpha{}^{\mu\nu}T^\beta{}_{\mu\nu}\right) \\ \quad + \frac{1}{3}N_\alpha{}^{\lambda\mu\nu}\left(2R^\beta{}_{\lambda\mu\nu} + \nabla_\lambda T^\beta{}_{\mu\nu} + \frac{1}{2}T^\sigma{}_{\mu\nu}T^\beta{}_{\lambda\sigma}\right) - N_\kappa{}^{\beta\mu\nu}R^\kappa{}_{\alpha\mu\nu} \equiv -I_\alpha{}^\beta; \\ M_\alpha{}^{(\beta\gamma)} + \overset{*}{\nabla}_\mu N_\alpha{}^{\beta\gamma\mu} + N_\alpha{}^{(\beta|\mu\nu}T^{|\gamma)}{}_{\mu\nu} \equiv 0; \\ N_\alpha{}^{(\beta\gamma\delta)} \equiv 0, \end{cases}$$

(16)

(17)

(18)

(19)

where

$$I_\alpha \overset{def}{=} \frac{\Delta I}{\Delta\Phi^A}\Phi_\alpha|^A; \qquad I_\alpha{}^\beta \overset{def}{=} \frac{\Delta I}{\Delta\Phi^A}\Phi_\alpha{}^\beta|^A. \tag{20}$$

Originally the system analogous to the above has been obtained in a non-covariant form by Klein [9] for purely metric theories of gravity. Therefore we will name system (16)–(19) as the *Klein identities.*

Statement 1 *The Klein identities* (16)–(19) *present complete manifestly covariant universal system of differential identities, which is valid in an arbitrary diffeomorfically invariant field theory.*

Subtracting the divergence $\overset{*}{\nabla}_\beta$ of (17) from the identity (16) we obtain the new identity

$$\overset{*}{\nabla}_\mu I_\alpha{}^\mu - I_\alpha \equiv 0 \tag{21}$$

that is the *Noether identity* rewritten in a manifestly covariant form.

Statement 2 *The (usual) Noether identity* (21) *is a consequence of the Klein identities* (16)–(19).

Analyzing the system (16)–(19), one shows that the following is valid:

Statement 3 *Instead of the Klein system* (16)–(19), *one can use the equivalent Klein-Noether system of identities:*

$$\begin{cases} \overset{*}{\nabla}_\mu I_\alpha{}^\mu \equiv I_\alpha; & (22) \\ \left(U_\alpha{}^\beta - \frac{1}{3}N_\lambda{}^{\beta\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\right) + \overset{*}{\nabla}_\mu\left(M_\alpha{}^{[\beta\mu]} - \frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\beta\mu]} + \frac{1}{3}N_\alpha{}^{[\beta|\rho\sigma}T^{|\mu]}{}_{\rho\sigma}\right) & \\ \quad + \frac{1}{2}\left(M_\alpha{}^{[\rho\sigma]} - \frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\rho\sigma]} + \frac{1}{3}N_\alpha{}^{[\rho|\kappa\lambda}T^{|\sigma]}{}_{\kappa\lambda}\right)T^\beta{}_{\rho\sigma} \equiv -I_\alpha{}^\beta; & (23) \\ M_\alpha{}^{(\beta\gamma)} + \overset{*}{\nabla}_\mu N_\alpha{}^{\beta\gamma\mu} + N_\alpha{}^{(\beta|\mu\nu}T^{|\gamma)}{}_{\mu\nu} \equiv 0; & (24) \\ N_\alpha{}^{(\beta\gamma\delta)} \equiv 0. & (25) \end{cases}$$

C. The Generalized Noether Superpotential. The Boundary Klein-Noether Theorem

Substituting $\{\delta_\xi \Phi^A\}$ (6) and $\{I_\alpha\}$ from (22) into the general Noether identity (7), one obtains another identity

$$\overset{*}{\nabla}_\mu \mathscr{J}^\mu[\delta\boldsymbol{\xi}] \equiv 0, \tag{26}$$

which has a meaning of the continuity equation for the current defined as $\mathscr{J}[\delta\boldsymbol{\xi}] \overset{def}{=} \{\mathscr{J}^\mu[\delta\boldsymbol{\xi}]\}$, where

$$\begin{aligned}\mathscr{J}^\mu[\delta\boldsymbol{\xi}] \overset{def}{=} & \left(U_\alpha{}^\mu + I_\alpha{}^\mu\right)\delta\xi^\alpha \\ & + M_\alpha{}^{\beta\mu}\nabla_\beta\,\delta\xi^\alpha + N_\alpha{}^{\beta\gamma\mu}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha.\end{aligned} \tag{27}$$

Note that the identity (26) takes a place *independently of equations of motion hold.*

Statement 4 (Covariant Poincaré Lemma) *In an arbitrary Riemann-Cartan space an identically conserved current $\mathscr{J}[\delta\boldsymbol{\xi}]$ can be represented locally in the form*

$$\mathscr{J}^\mu[\delta\boldsymbol{\xi}] = \overset{*}{\nabla}_\nu \theta^{\mu\nu}[\delta\boldsymbol{\xi}] + \frac{1}{2}\theta^{\rho\sigma}[\delta\boldsymbol{\xi}]T^\mu{}_{\rho\sigma} \tag{28}$$

where

$$\boldsymbol{\theta}[\delta\boldsymbol{\xi}] \overset{def}{=} \{\theta^{\mu\nu}[\delta\boldsymbol{\xi}]\}; \qquad \theta^{[\mu\nu]}[\delta\boldsymbol{\xi}] = \theta^{\mu\nu}[\delta\boldsymbol{\xi}] \tag{29}$$

is an antisymmetric tensor — the generalized Noether superpotential.

Statement 5 *For the current $\mathscr{J}[\delta\boldsymbol{\xi}]$ (27) the generalized Noether*

superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (29) *has the form*

$$\begin{aligned}&\theta^{\mu\nu}[\delta\boldsymbol{\xi}]\\&=\left\{-M_\alpha{}^{[\mu\nu]}+\frac{2}{3}\left(\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\mu\nu]}+\frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_\alpha{}^{\nu]\rho\sigma}\right)\right\}\delta\xi^\alpha\\&+\left\{-\frac{4}{3}N_\alpha{}^{\beta[\mu\nu]}\right\}\nabla_\beta\delta\xi^\alpha.\end{aligned}\tag{30}$$

It is evidently that the current $\mathscr{J}[\delta\boldsymbol{\xi}]$ is connected with the generalized Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (12) by the relation:

$$J^\mu[\delta\boldsymbol{\xi}]=-I_\alpha{}^\mu\delta\xi^\alpha+\mathscr{J}^\mu[\delta\boldsymbol{\xi}].\tag{31}$$

Substituting the expression (28) into formula (31), integrating it over Σ and using the Stockes rule, we rewrite the generalized Noether charge (9) in the form

$$Q[\delta\boldsymbol{\xi};\Sigma]=-\int\limits_\Sigma d\sigma_\mu\, I_\alpha{}^\mu\delta\xi^\alpha+\frac{1}{2!}\oint\limits_{\partial\Sigma}ds_{\mu\nu}\,\theta^{\mu\nu}[\delta\boldsymbol{\xi}].\tag{32}$$

The above relation is a special case (in an integral form) of a more general

Statement 6 (The boundary Klein-Noether theorem) *In an arbitrary gauge-invariant theory the Noether current is presented by a sum of two terms, the first vanishes on equations of motion, the second is a divergence of a superpotential.*

D. The Generalized Symmetrized Noether Current

From now we call $\mathbf{J}[\delta\boldsymbol{\xi}]$ (12) and $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (30) as the generalized *canonical* Noether current and the generalized *canonical* superpotential, respectively. Recall that $\mathbf{J}[\delta\boldsymbol{\xi}]$ contains derivatives of a displacement vector $\boldsymbol{\nabla}\delta\boldsymbol{\xi}$, $\boldsymbol{\nabla}\boldsymbol{\nabla}\delta\boldsymbol{\xi}$.

We construct a new current $\overset{sym}{J}{}^{\mu}[\delta\boldsymbol{\xi}]$, instead of the canonical one $\mathbf{J}[\delta\boldsymbol{\xi}]$, with the property that it does not contain derivatives of $\delta\boldsymbol{\xi}$. In other words, we search for

$$\overset{sym}{J}{}^{\mu}[\delta\boldsymbol{\xi}] = \overset{sym}{U}{}_{\alpha}{}^{\mu}\delta\xi^{\alpha}. \tag{33}$$

Because a new current has to be also differentially conserved, we construct it by adding an antisymmetrical tensor $\mathscr{B}[\delta\boldsymbol{\xi}] \overset{def}{=} \left\{\mathscr{B}^{[\mu\nu]}[\delta\boldsymbol{\xi}] = \mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}]\right\}$, similarly to (28):

$$\overset{sym}{J}{}^{\mu}[\delta\boldsymbol{\xi}] \overset{def}{=} J^{\mu}[\delta\boldsymbol{\xi}] - \left(\overset{*}{\nabla}_{\nu}\mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] + \frac{1}{2}\mathscr{B}^{\rho\sigma}[\delta\boldsymbol{\xi}]T^{\mu}{}_{\rho\sigma}\right). \tag{34}$$

We call this formula as a *generalized Belinfante relation*, and a tensor $\mathscr{B}[\delta\boldsymbol{\xi}]$ — as a *generalized Belinfante tensor*. Calculations show that

Statement 7 *The generalized Belinfante tensor $\mathscr{B}[\delta\boldsymbol{\xi}]$ coincides with the generalized canonical superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$* (30)

$$\mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] = \theta^{\mu\nu}[\delta\boldsymbol{\xi}]; \tag{35}$$

the tensor $\overset{sym}{\mathbf{U}} \overset{def}{=} \{\overset{sym}{U}{}_{\alpha}{}^{\mu}\}$ has the form

$$\begin{aligned}
&\overset{sym}{U}{}_{\alpha}{}^{\mu} = \left(U_{\alpha}{}^{\mu} - \frac{1}{3}N_{\lambda}{}^{\mu\rho\sigma}R^{\lambda}{}_{\alpha\rho\sigma}\right) \\
&+\overset{*}{\nabla}_{\nu}\left[M_{\alpha}{}^{[\mu\nu]} - \frac{2}{3}\left(\overset{*}{\nabla}_{\lambda}N_{\alpha}{}^{\lambda[\mu\nu]} + \frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_{\alpha}{}^{\nu]\rho\sigma}\right)\right] \\
&+\frac{1}{2}\left[M_{\alpha}{}^{[\rho\sigma]} - \frac{2}{3}\left(\overset{*}{\nabla}_{\lambda}N_{\alpha}{}^{\lambda[\rho\sigma]} + \frac{1}{2}T^{[\rho}{}_{\varepsilon\kappa}N_{\alpha}{}^{\sigma]\varepsilon\kappa}\right)\right]T^{\mu}{}_{\rho\sigma}.
\end{aligned} \tag{36}$$

Note that the right hand side of (36) exactly coincides with the left hand side of the Klein identity (23). Therefore, one can write also

$$\overset{sym}{U}{}_{\alpha}{}^{\mu} = -I_{\alpha}{}^{\mu}. \tag{37}$$

It turns out that in the manifestly generally covariant theories (see Sec. III) the tensor $\overset{sym}{\mathbf{U}}$ coincides with symmetrized energy-momentum tensor $\overset{sym}{\mathbf{t}} \overset{def}{=} \{\overset{sym}{t}{}^{\mu}{}_{\alpha}\}$:

$$\overset{sym}{U}{}_{\alpha}{}^{\mu} = \overset{sym}{t}{}^{\mu}{}_{\alpha}. \tag{38}$$

Then relations (37), (38) are a proof of the

Statement 8 *The symmetrized energy-momentum tensor* $\overset{sym}{\mathbf{t}}$ *does not depend on divergences in the Lagrangian.*

III. MANIFESTLY GENERALLY COVARIANT THEORIES

A. Structure of Lagrangians

In Sec. II the quantities and relations in the theories of the most general type have been constructed. In the present section, we concretize them. We apply the developed formalism for the study of manifestly generally covariant theories, and all the statements, here, are related to them. We call a theory as *manifestly generally covariant* if its Lagrangian $\mathscr{L}$ is a generally covariant scalar constructed as algebraic scalar function of manifestly covariant objects, which are transformed following the linear *homogeneous* representations of the diffeomorphism group. This means that $\mathscr{L}$ is an algebraic function of scalar contractions of tensor (and/or spinor) field functions and their covariant derivatives; besides manifest dependence on field variables, $\mathscr{L}$ can also depend on curvature and torsion tensors independently. It seems that almost all the physically interesting theories are manifestly generally covariant or can be presented in such a form.

Consider field theories presented by the action (1), with Lagrangians in a manifestly covariant form:

$$\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R};\ \mathbf{T}, \nabla\mathbf{T}, \nabla\nabla\mathbf{T};\ \varphi, \nabla\varphi, \nabla\nabla\varphi). \tag{39}$$

Here, the total set of the fields Φ is presented by the metric tensor $\mathbf{g}$, by the torsion tensor $\mathbf{T}$ and by a set of the matter fields $\varphi \stackrel{def}{=} \{\varphi^a(x);\ a = \overline{1,n}\}$, which are considered as tensorial ones also. Lagrangians of the type (39) include, together with the minimal coupling, the non-minimal coupling related both to the curvature and to the torsion. In this section we present relations and conserved quantities (currents and superpotentials) constructed in the Sec. II in a maximally concrete form that follows from the concrete structure of the Lagrangian (39).

Because the fields $\mathbf{T}$ and φ are included in the Lagrangian in a similar way, for simplification of notations we unite them into the unique set ϕ:

$$\mathbf{T}, \varphi \quad \rightarrow \quad \phi \stackrel{def}{=} \{\phi^a\} \stackrel{def}{=} \{\mathbf{T}, \varphi\}. \tag{40}$$

Now, the Lagrangian (39) is presented as

$$\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R};\ \phi, \nabla\phi, \nabla\nabla\phi). \tag{41}$$

One has to keep in mind that the torsion $\mathbf{T}$ is included in the Lagrangian (39) not only *explicitly* as arguments $\mathbf{T}$, $\nabla\mathbf{T}$ and $\nabla\nabla\mathbf{T}$, but *not explicitly* also over the connection Γ, which is used for constructing the covariant derivative ∇ and the curvature tensor $\mathbf{R}$.

B. Structure of the Tensors U, M, N. Canonical EMT t and ST s

Direct calculations show that in the case of the Lagrangian of the type (41) the tensors **U** (13), **M** (14) and **N** (15) have the form

$$\begin{cases} U_\alpha{}^\mu = t^\mu{}_\alpha + \left(\Delta^{\overline{\mu\beta\gamma}}_{\pi\rho\sigma} s^{\pi,\rho\sigma}\right) T_{\gamma,\beta\alpha} + \frac{1}{2} G^{\beta\gamma\varepsilon\mu} R_{\alpha\beta\gamma\varepsilon} + \left[\overset{*}{\nabla}_\nu \left(G^{\beta\gamma\mu\nu} T_{\beta,\gamma\alpha}\right) + \frac{1}{2}\left(G^{\beta\gamma\rho\sigma} T_{\beta,\gamma\alpha}\right) T^\mu{}_{\rho\sigma}\right] \\ M_\kappa{}^{\lambda\mu} = -\left(\Delta^{\overline{\mu\lambda\alpha}}_{\pi\rho\sigma} s^{\pi,\rho\sigma}\right) g_{\alpha\kappa} - \left(\overset{*}{\nabla}_\nu G_\kappa{}^{\lambda\mu\nu} + \frac{1}{2} G_\kappa{}^{\lambda\rho\sigma} T^\mu{}_{\rho\sigma}\right) + \frac{1}{2} G_\kappa{}^{\alpha\beta\mu} T^\lambda{}_{\alpha\beta} - \left(G_\alpha{}^{\beta\lambda\mu} T^\alpha{}_{\beta\kappa}\right); \\ N_\alpha{}^{\beta\gamma\mu} = G_\alpha{}^{(\beta\gamma)\mu}. \end{cases} \tag{42}$$

(43)

(44)

Here,

$$G_\kappa{}^{\lambda\mu\nu} \overset{def}{=} 2\frac{\partial \mathscr{L}}{\partial R^\kappa{}_{\lambda\mu\nu}}; \tag{45}$$

$$\Delta^{\overline{\alpha\beta\gamma}}_{\lambda\mu\nu} \overset{def}{=} \frac{1}{2}\left(\delta^\beta_\lambda \delta^\alpha_\mu \delta^\gamma_\nu + \delta^\gamma_\lambda \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\lambda \delta^\beta_\mu \delta^\gamma_\nu\right); \tag{46}$$

$$t^\mu{}_\alpha \overset{def}{=} \mathscr{L}\delta^\mu_\alpha - \frac{\Delta I}{\Delta(\nabla_\mu \phi^a)} \nabla_\alpha \phi^a - \frac{\partial \mathscr{L}}{\partial(\nabla_\mu \nabla_\nu \phi^a)} \nabla_\alpha \nabla_\nu \phi^a - G^{\beta\gamma\varepsilon\mu} R_{\beta\gamma\varepsilon\alpha}; \tag{47}$$

$$\begin{aligned} s^\pi{}_{\rho\sigma} \overset{def}{=} & -2\left(\overset{*}{\nabla}_\eta G_{\rho\sigma}{}^{\pi\eta} + \frac{1}{2} G_{\rho\sigma}{}^{\alpha\beta} T^\pi{}_{\alpha\beta}\right) \\ & +2\left(\frac{\Delta I}{\Delta(\nabla_\pi \phi^a)} (\Delta_{[\rho\sigma]})^a{}_b \phi^b + \frac{\partial \mathscr{L}}{\partial(\nabla_\pi \nabla_\alpha \phi^a)} \left[(\Delta_{[\rho\sigma]})^a{}_b \nabla_\alpha \phi^b - g_{\alpha[\rho} \nabla_{\sigma]} \phi^a\right]\right); \end{aligned} \tag{48}$$

$$\frac{\Delta I}{\Delta(\nabla_\mu \phi^a)} \overset{def}{=} \frac{\partial \mathscr{L}}{\partial(\nabla_\mu \phi^a)} - \overset{*}{\nabla}_\nu \left(\frac{\partial \mathscr{L}}{\partial(\nabla_\nu \nabla_\mu \phi^a)}\right) \tag{49}$$

and $\{(\Delta_{\rho\sigma})^a{}_b\}$ are *the Belinfante Rosenfeld symbols* – some combinations of the Kronecker δ's (Explicit expressions are given in Appendix C.1 of

Ref. [2]).

Notice also that the tensors $\mathbf{t} \stackrel{def}{=} \{t^{\mu}{}_{\nu}\}$ (47) and $\mathbf{s} \stackrel{def}{=} \{s^{\pi}{}_{\rho\sigma}\}$ (48) are just the *generalized canonical energy-momentum tensor* (EMT) and *generalized canonical spin tensor* (ST), corresponding to the Lagrangian (39). The reason for this statement is that basing on the above definitions of the EMT and ST, one obtains the standard equations of balance for the EMT. Besides, as we will see in the Sec. IV, the gravitational field equations acquire the form, naturally generalizing the gravitational field equations of the Einstein-Cartan theory.

It is worse to note that the sequence of the second derivative in the multiplier $\{\nabla_{\alpha}\nabla_{\nu}\phi^{a}\}$ in (47) is *reverse* to the sequence that follows from the construction of the canonical EMT by the direct application of the 1-st Noether theorem. The last term in (47) as well as the items in the first parentheses on the right hand side of (48) have appeared due to the non-minimal coupling with the metric field. These items *in principal* cannot be obtained with the use of the 1-st Noether theorem in the Minkowski space and covariantization of the expressions.

From (44) and (45) directly follows

Statement 9 *The tensor* $\mathbf{N}$ (44) *is not equal to zero only if the Lagrangian contains explicitly the curvature tensor* $\mathbf{R}$.

Next, the substitution of the expressions (42) (44) into the formula (12) leads to

Statement 10 *The canonical current* $\mathbf{J}[\delta\boldsymbol{\xi}]$ (12) *essentially is constructed,*

basing on the canonical dynamic quantities $\mathbf{t}$ (47), $\mathbf{s}$ (48) *and the tensor* $\mathbf{G}$ (45).

Also, one has

Statement 11 *The symmetrized Noether current* $\overset{sym}{\mathbf{J}}\ [\delta\boldsymbol{\xi}]$ *is expressed through only the symmetrized EMT* $\overset{sym}{\mathbf{t}}$ *even in the case of the Lagrangian of the most general type* (39)*:*

$$\overset{sym}{J}{}^{\mu}[\delta\boldsymbol{\xi}] \overset{def}{=} \overset{sym}{t}{}^{\mu}{}_{\alpha}\delta\xi^{\alpha}. \tag{50}$$

C. Structure of the Variational Derivatives. Modified Canonical ST $\overset{mod}{\mathbf{s}}$

In the work [2] we have shown the general

Statement 12 (Structure of the $\Delta I/\Delta\mathbf{T}$) *Variational derivative* $\Delta I/\Delta\mathbf{T}$ *of the action functional* I *with respect to the torsion tensor* $\mathbf{T}$ *is equal to*

$$\frac{\Delta I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} = \frac{\Delta^{*} I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} + \frac{1}{2} b^{\gamma\beta}{}_{\varepsilon}. \tag{51}$$

where

$$\begin{aligned} &\frac{\Delta^{*} I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} \overset{def}{=} \frac{\partial^{*}\mathscr{L}}{\partial T^{\varepsilon}{}_{\beta\gamma}} - \overset{*}{\nabla}_{\mu}\left(\frac{\partial\mathscr{L}}{\partial(\nabla_{\mu}T^{\varepsilon}{}_{\beta\gamma})}\right) \\ &+\overset{*}{\nabla}_{\nu}\overset{*}{\nabla}_{\mu}\left(\frac{\partial\mathscr{L}}{\partial(\nabla_{\mu}\nabla_{\nu}T^{\varepsilon}{}_{\beta\gamma})}\right); \\ &b^{\gamma\beta\alpha} \overset{def}{=} \Delta^{\overline{\gamma\beta}\alpha}_{\pi\rho\sigma} s^{\pi,\rho\sigma}; \qquad b^{[\gamma\beta]\alpha} = b^{\gamma\beta\alpha}. \end{aligned} \tag{52}$$

The Belinfante tensor: $\mathbf{b} \overset{def}{=} \left\{b^{\gamma\beta\alpha}\right\}$ *is induced by the ST* $\mathbf{s}$.

Consequence 1 *In the case of only minimal* **T***-coupling (when the Lagrangian* $\mathscr{L}$ *does not contain the torsion tensor* **T** *explicitly) one has*

$$\frac{\Delta I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} = \frac{1}{2} b^{\gamma\beta}{}_{\varepsilon}. \tag{53}$$

Earlier, the same result (53) has been proved only for the Lagrangians of the type $\mathscr{L} = \mathscr{L}(\mathbf{g}; \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi})$ with a more simple presentations both of the ST and of the Belinfante tensor (see Refs. [10–16]). We have proved a more general claim: *the formula* (53) *is left valid for the Lagrangians of a more general type* $\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R}; \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi})$.

The formula (51) *shows that the presence of a non-minimal coupling with torsion changes* (53). The requirement (the desire) to conserve a sense of the variational derivative (53) even at the presence of a non-minimal **T**-coupling leads to a necessity to modify both the initial Belinfante tensor and the initial ST. Let us demonstrate the modification step by step. Rewrite the formula (51) in the form of (53):

$$\frac{\Delta I}{\Delta T^{\varepsilon}{}_{\beta\gamma}} = \frac{1}{2} \overset{mod}{b}{}^{\gamma\beta}{}_{\varepsilon}, \tag{54}$$

where the modified Belinfante tensor $\overset{mod}{\mathbf{b}} = \{\overset{mod}{b}{}^{\gamma\beta\alpha}\}$ is defined analogously to the initial one (that is with the use of *any* ST):

$$\overset{mod}{b}{}^{\gamma\beta\alpha} \overset{def}{=} \Delta^{\overline{\gamma\beta\alpha}}_{\pi\rho\sigma} \overset{mod}{s}{}^{\pi,\rho\sigma}. \tag{55}$$

The *modified* Belinfante tensor and canonical ST can be represented as initial ones and correspondent additions:

$$\overset{mod}{b}{}^{\gamma\beta\alpha} \overset{def}{=} b^{\gamma\beta\alpha} + \overset{add}{b}{}^{\gamma\beta\alpha}; \tag{56}$$

$$\overset{mod}{s}{}^{\pi}{}_{\rho\sigma} \overset{def}{=} s^{\pi}{}_{\rho\sigma} + \overset{add}{s}{}^{\pi}{}_{\rho\sigma}. \tag{57}$$

Finally, combining (51) - (55), one obtains the definitions for the additional Belinfante tensor and ST:

$$\overset{add}{b}{}^{\gamma\beta}{}_{\varepsilon} = 2\frac{\Delta^* I}{\Delta T^{\varepsilon}{}_{\beta\gamma}}; \tag{58}$$

$$\overset{add}{S}{}^{\pi,\rho\sigma} = -4g^{[\sigma|\varepsilon}\frac{\Delta^* I}{\Delta T^{\varepsilon}{}_{|\rho]\pi}}. \tag{59}$$

The *metric EMT* $\overset{met}{\mathbf{t}} \overset{def}{=} \{\overset{met}{t}{}^{\beta\gamma}\}$ is defined by a standard way

$$\frac{1}{2}\overset{met}{t}{}^{\beta\gamma} \overset{def}{=} \frac{\Delta I}{\Delta g_{\beta\gamma}}. \tag{60}$$

D. Generalization of the Belinfante Symmetrization Procedure. Symmetrized EMT $\overset{sym}{\mathbf{t}}$

In a Riemann-Cartan space the Belinfante symmetrization procedure is generalized in a non-trivial way.

Statement 13 *In an arbitrary Riemann-Cartan space the Belinfante symmetrized EMT* $\overset{sym}{\mathbf{t}} \overset{def}{=} \{\overset{sym}{t}{}^{\mu}{}_{\nu}\}$ *must be constructed by the rule*

$$\begin{aligned}\overset{sym}{t}{}^{\mu}{}_{\nu} &\overset{def}{=} t^{\mu}{}_{\nu} \\ &+ \left[\overset{*}{\nabla}_{\lambda} b^{\mu\lambda}{}_{\nu} + \frac{1}{2}b^{\rho\sigma}{}_{\nu}T^{\mu}{}_{\rho\sigma} + b^{\mu\beta}{}_{\alpha}T^{\alpha}{}_{\beta\nu}\right].\end{aligned} \tag{61}$$

E. The Physical Sense of the Klein and Noether Identities. Equations of balance. Modified Canonical EMT $\overset{mod}{\mathbf{t}}$

Let us examine a physical sense of the identities by Noether and Klein. Consequent substitutions of the expressions for the tensors **U** (42), **M** (43),

N (44) into the identities (22), (24), (25), (23) and (16) permit to prove the next statements:

Statement 14 (The physical sense of the Noether identity) *The Noether identity* (22) *has the explicit form*

$$\overset{*}{\nabla}_\mu \overset{met}{t}{}^{\mu}{}_{\nu} \equiv - \overset{met}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \left\{ \overset{*}{\nabla}_\mu \left[\frac{\Delta I}{\Delta \phi^a} (\Delta^\mu{}_\nu)^{\,a}|_b \, \phi^b \right] + \frac{\Delta I}{\Delta \phi^a} \left[\nabla_\nu \phi^a + (\Delta^\mu{}_\lambda)^{\,a}|_b \, \phi^b \, T^{\lambda}{}_{\mu\nu} \right] \right\} \tag{62}$$

or (in an expanded presentation)

$$\begin{aligned} \overset{*}{\nabla}_\mu \left(\overset{met}{t}{}^{\mu}{}_{\nu} + \overset{*}{\nabla}_\eta \overset{mod}{b}{}^{\eta\mu}{}_{\nu} \right) \equiv & - \left(\overset{met}{t}{}^{\mu}{}_{\lambda} + \overset{*}{\nabla}_\eta \overset{mod}{b}{}^{\eta\mu}{}_{\lambda} \right) T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{mod}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} \\ & + \left\{ \overset{*}{\nabla}_\mu \left[\frac{\Delta I}{\Delta \varphi^a} (\Delta^\mu{}_\nu)^{\,a}|_b \, \varphi^b \right] + \frac{\Delta I}{\Delta \varphi^a} \left[\nabla_\nu \varphi^a + (\Delta^\mu{}_\lambda)^{\,a}|_b \, \varphi^b \, T^{\lambda}{}_{\mu\nu} \right] \right\} \end{aligned} \tag{63}$$

and is the basis for defining the equations of balance for the metric EMT:

$$\overset{*}{\nabla}_\mu \overset{met}{t}{}^{\mu}{}_{\nu} = - \overset{met}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} \qquad \textit{(on the } \phi\textit{-equations)} \tag{64}$$

or

$$\overset{*}{\nabla}_\mu \left(\overset{met}{t}{}^{\mu}{}_{\nu} + \overset{*}{\nabla}_\eta \overset{mod}{b}{}^{\eta\mu}{}_{\nu} \right) = - \left(\overset{met}{t}{}^{\mu}{}_{\lambda} + \overset{*}{\nabla}_\eta \overset{mod}{b}{}^{\eta\mu}{}_{\lambda} \right) T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{mod}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} \qquad \textit{(on the } \varphi\textit{-equ} \tag{65}$$

Statement 15 (On the 4-th and 3-rd Klein identities) *The 4-th and 3-rd Klein identities* (25) *and* (24) *are satisfied automatically.*

Statement 16 (The physical sense of the 2-nd Klein identity) *The 2-nd Klein identity* (23) *has the explicit form*

$$\overset{sym}{t}{}^{\mu}{}_{\nu} \equiv \overset{met}{t}{}^{\mu}{}_{\nu} - \frac{\Delta I}{\Delta \phi^a} (\Delta^\mu{}_\nu)^{\,a}|_b \, \phi^b \tag{66}$$

or

$$t^{\mu}{}_{\nu} - \overset{*}{\nabla}_{\lambda} b^{\lambda\mu}{}_{\nu} \equiv \overset{met}{t}{}^{\mu}{}_{\nu} + \left(\frac{1}{2} \overset{add}{b}{}^{\rho\sigma}{}_{\nu} T^{\mu}{}_{\rho\sigma} + \overset{add}{b}{}^{\mu\beta}{}_{\alpha} T^{\alpha}{}_{\beta\nu} \right) - \frac{\Delta I}{\Delta \varphi^{a}} (\Delta^{\mu}{}_{\nu})^{\,a}|_{b}\, \varphi^{b} \tag{67}$$

and claims that on the ϕ-equations symmetrized EMT $\overset{sym}{\mathbf{t}}$ (61) *is equal to the metric EMT* $\overset{met}{\mathbf{t}}$ (60)*:*

$$\overset{sym}{t}{}^{\mu}{}_{\nu} = \overset{met}{t}{}^{\mu}{}_{\nu} \qquad \textit{(on the } \phi\textit{-equations)} \tag{68}$$

or

$$t^{\mu}{}_{\nu} - \overset{*}{\nabla}_{\lambda} b^{\lambda\mu}{}_{\nu} = \overset{met}{t}{}^{\mu}{}_{\nu} + \left(\frac{1}{2} \overset{add}{b}{}^{\rho\sigma}{}_{\nu} T^{\mu}{}_{\rho\sigma} + \overset{add}{b}{}^{\mu\beta}{}_{\alpha} T^{\alpha}{}_{\beta\nu} \right) \qquad \textit{(on the } \varphi\textit{-equations).} \tag{69}$$

Statement 17 (The physical sense of the 1-st Klein identity) *The 1-st Klein identity* (16) *has the explicit form*

$$\overset{*}{\nabla}_{\mu} t^{\mu}{}_{\nu} \equiv -t^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} s^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} - \frac{1}{2} b^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} + \frac{\Delta I}{\Delta \phi^{a}} \nabla_{\nu} \phi^{a} \tag{70}$$

or

$$\overset{*}{\nabla}_{\mu} t^{\mu}{}_{\nu} \equiv -t^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} s^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} - \frac{1}{2} \overset{add}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} + \frac{\Delta I}{\Delta \varphi^{a}} \nabla_{\nu} \varphi^{a} \tag{71}$$

and is the basis for constructing the equations of balance for the canonical EMT $\mathbf{t}$

$$\overset{*}{\nabla}_{\mu} t^{\mu}{}_{\nu} = -t^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} s^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} - \frac{1}{2} b^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} \qquad \textit{(on the } \phi\textit{-equations)} \tag{72}$$

or

$$\overset{*}{\nabla}_{\mu} t^{\mu}{}_{\nu} = -t^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} s^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} + \frac{1}{2} \overset{add}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} \qquad \textit{(on the } \varphi\textit{-equations).} \tag{73}$$

In Refs. [10–16], for the Lagrangian of the type $\mathscr{L} = \mathscr{L}(\mathbf{g};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi})$ the equation of balance for the canonical EMT

$$\overset{*}{\nabla}_{\mu} t^{\mu}{}_{\nu} = -t^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} s^{\pi,\rho\sigma} R_{\rho\sigma\pi\nu} \qquad \text{(on the } \boldsymbol{\varphi}\text{-equations)} \tag{74}$$

has been obtained. The result (74) is left valid in a more general case also, when the Lagrangian has a form: $\mathscr{L} = \mathscr{L}(\mathbf{g}, \mathbf{R};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi})$ because the last term in (73) does not appear. In the case of non-minimal $\mathbf{T}$-coupling the right hand side of (73) contains *additional* term $\left(\frac{1}{2} \overset{add}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma}\right)$. However, the *new* equation (73) can be also transformed to the form (74), see for details [2]. Only, the EMT and ST in (74) are changed to modified ones. The modification of the EMT is analogous to the modification of the canonical ST in (57) and (59). The *modified canonical EMT* $\overset{mod}{\mathbf{t}} \overset{def}{=} \{\overset{mod}{t}{}^{\mu}{}_{\nu}\}$:

$$\overset{mod}{t}{}^{\mu}{}_{\nu} \overset{def}{=} t^{\mu}{}_{\nu} + \overset{add}{t}{}^{\mu}{}_{\nu}, \tag{75}$$

where the *additional EMT* $\overset{add}{\mathbf{t}} \overset{def}{=} \{\overset{add}{t}{}^{\mu}{}_{\nu}\}$ is defined as

$$-\overset{add}{t}{}^{\mu}{}_{\nu} \overset{def}{=} \overset{*}{\nabla}_{\lambda} \overset{add}{b}{}^{\mu\lambda}{}_{\nu} + \frac{1}{2} \overset{add}{b}{}^{\kappa\lambda}{}_{\nu} T^{\mu}{}_{\kappa\lambda} + \overset{add}{b}{}^{\mu\lambda}{}_{\kappa} T^{\kappa}{}_{\lambda\nu}. \tag{76}$$

It is evidently that the EMT $\overset{mod}{\mathbf{t}}$ (75) in the case of minimal $\mathbf{T}$-coupling only transforms to the usual canonical EMT $\mathbf{t}$. By the definition (76), The Belinfante symmetrization of the type (61) applied to $\overset{add}{\mathbf{t}}$ leads to zero identically. Therefore,

Statement 18 *The symmetrized EMT* $\overset{sym}{\mathbf{t}}$ *constructed by the*

symmetrization of $\overset{mod}{\mathbf{t}}$ (75) *with the use of* $\overset{mod}{\mathbf{b}}$ (55) *by the rule*

$$\overset{sym}{t}{}^{\mu}{}_{\nu} = \overset{mod}{t}{}^{\mu}{}_{\nu} + \left[\overset{*}{\nabla}_{\lambda} \overset{mod}{b}{}^{\mu\lambda}{}_{\nu} + \frac{1}{2} \overset{mod}{b}{}^{\kappa\lambda}{}_{\nu} T^{\mu}{}_{\kappa\lambda} + \overset{mod}{b}{}^{\mu\lambda}{}_{\kappa} T^{\kappa}{}_{\lambda\nu}\right] \tag{77}$$

exactly coincides with the (usual) symmetrized EMT $\overset{sym}{\mathbf{t}}$ (61).

Statement 19 *In the terms of the modified canonical EMT* $\overset{mod}{\mathbf{t}}$ *the 1-st Klein identities* (70) *or* (71) *can be rewritten as*

$$\overset{*}{\nabla}_{\mu} \overset{mod}{t}{}^{\mu}{}_{\nu} \equiv - \overset{mod}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{mod}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} - \frac{1}{2} \overset{mod}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} + \frac{\Delta I}{\Delta \phi^{a}} \nabla_{\nu} \phi^{a} \tag{78}$$

or

$$\overset{*}{\nabla}_{\mu} \overset{mod}{t}{}^{\mu}{}_{\nu} \equiv - \overset{mod}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{mod}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} + \frac{\Delta I}{\Delta \varphi^{a}} \nabla_{\nu} \varphi^{a}. \tag{79}$$

They are the basis for the equations of balance for the modified canonical EMT $\overset{mod}{\mathbf{t}}$ *:*

$$\overset{*}{\nabla}_{\mu} \overset{mod}{t}{}^{\mu}{}_{\nu} = - \overset{mod}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{mod}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} - \frac{1}{2} \overset{mod}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} \quad \textit{(on the } \phi\textit{-equations)} \tag{80}$$

or

$$\overset{*}{\nabla}_{\mu} \overset{mod}{t}{}^{\mu}{}_{\nu} = - \overset{mod}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{1}{2} \overset{mod}{s}{}^{\pi}{}_{\rho\sigma} R^{\rho\sigma}{}_{\pi\nu} \tag{81}$$

(on the φ*-equations).*

Now, the equation (81) has the same structure as the equation (74). Also notice that, when the equations for the torsion field $\Delta I/\Delta \mathbf{T} = 0$ hold, then

$$\overset{mod}{t}{}^{\mu}{}_{\nu} = \overset{sym}{t}{}^{\mu}{}_{\nu} \qquad \text{(on the T-equations)}, \tag{82}$$

as it follows from (77) and (54).

At last, let us find the identities and the equations of balance for the symmetrized EMT $\overset{sym}{\mathbf{t}}$. Use (61) for rewriting $\mathbf{t}$ as a function of $\overset{sym}{\mathbf{t}}$ and $\mathbf{b}$, substitute the result into (70) and (71) and find, respectively,

Statement 20 *The 1-st Klein identity, as well as the 2-nd one lead to the identities*

$$\overset{*}{\nabla}_{\mu} \overset{sym}{t}{}^{\mu}{}_{\nu} \equiv - \overset{sym}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} + \frac{\Delta I}{\Delta \phi^a} \nabla_{\nu} \phi^a \tag{83}$$

or

$$\begin{aligned} \overset{*}{\nabla}_{\mu} \overset{sym}{t}{}^{\mu}{}_{\nu} \equiv & - \overset{sym}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} \\ & + \frac{1}{2} \overset{mod}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} + \frac{\Delta I}{\Delta \varphi^a} \nabla_{\nu} \varphi^a, \end{aligned} \tag{84}$$

which are the basis for constructing the equations of balance for the symmetrized EMT $\overset{sym}{\mathbf{t}}$ *:*

$$\overset{*}{\nabla}_{\mu} \overset{sym}{t}{}^{\mu}{}_{\nu} = - \overset{sym}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} \text{ (on the } \phi\text{-equations)} \tag{85}$$

or

$$\begin{aligned} \overset{*}{\nabla}_{\mu} \overset{sym}{t}{}^{\mu}{}_{\nu} = & - \overset{sym}{t}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\mu\nu} \\ & + \frac{1}{2} \overset{mod}{b}{}^{\gamma\beta}{}_{\alpha} \nabla_{\nu} T^{\alpha}{}_{\beta\gamma} \text{ (on the } \varphi\text{-equations).} \end{aligned} \tag{86}$$

F. Explicit Form of the Superpotential $\theta[\delta\xi]$

Let us turn to the superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$. Substituting the explicit expressions for the tensors **U** (42), **M** (43) and **N** (44) into the formula (30) we obtain

Statement 21 *In manifestly generally covariant field theories with Lagrangians of type* (39) *the generalized superpotential* $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ *has the explicit form*

$$\begin{aligned}\theta^{\mu\nu}[\delta\boldsymbol{\xi}] \stackrel{def}{=} &\left[-b^{\mu\nu}{}_{\alpha} + G_{\kappa}{}^{\lambda\mu\nu}T^{\kappa}{}_{\lambda\alpha}\right]\delta\xi^{\alpha} \\ &+ \left[-G_{\alpha}{}^{\beta\mu\nu}\right]\nabla_{\beta}\delta\xi^{\alpha},\end{aligned} \tag{87}$$

i.e. it is expressed through only the Belinfante tensor **b** *induced by the* canonical ST **s** *and the tensor* **G**.

IV. STRUCTURE AND INTERPRETATION OF THE EQUATIONS OF GRAVITATIONAL FIELD

A. Splitting of the Total Lagrangian

Represent the total Lagrangian (39) as a sum of the pure gravitational $\mathscr{L}^{G}$ and matter $\mathscr{L}^{M}$ parts:

$$\begin{aligned}\mathscr{L} &= \mathscr{L}(\mathbf{g}, \mathbf{R};\ \mathbf{T}, \boldsymbol{\nabla}\mathbf{T}, \boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{T};\ \boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\varphi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varphi}) \\ &\stackrel{def}{=} \mathscr{L}(\mathbf{g}, \mathbf{R};\ \boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\phi}) = \mathscr{L}^{G} + \mathscr{L}^{M},\end{aligned} \tag{88}$$

where

$$\mathscr{L}^{G} = \mathscr{L}^{G}(\mathbf{g}, \mathbf{R}) \stackrel{def}{=} \mathscr{L}(\mathbf{g}, \mathbf{R}; 0, 0, 0); \tag{89}$$

$$\mathscr{L}^M = \mathscr{L}^M(\mathbf{g}, \mathbf{R};\ \phi, \nabla\phi, \nabla\nabla\phi) \stackrel{def}{=} \mathscr{L} - \mathscr{L}^G. \tag{90}$$

Remark that in spite of the Lagrangian $\mathscr{L}^G$ does not contain the torsion $\mathbf{T}$ *explicitly*, it contains the torsion *not explicitly* over the connection $\mathbf{\Gamma}$, which is used for constructing the curvature tensor $\mathbf{R}$. Therefore, the proposed splitting (88) - (90) is non-trivial.

It is evidently that the splitting of the Lagrangian (88) leads to a correspondent splitting of the action functional:

$$I = \int\limits_{\Sigma_1}^{\Sigma_2} dx\, \sqrt{-g}\mathscr{L} = \int\limits_{\Sigma_1}^{\Sigma_2} dx\, \sqrt{-g}\mathscr{L}^G + \int\limits_{\Sigma_1}^{\Sigma_2} dx\, \sqrt{-g}\mathscr{L}^M$$
$$\stackrel{def}{=} I^G + I^M.$$

Of course, the Lagrangian of the vacuum system $\mathscr{L}^G$ has to be *generally covariant scalar*, and then the matter Lagrangian $\mathscr{L}^M$ is, like this, also. Therefore all the above results and conclusions related to the total Lagrangian $\mathscr{L}$ are left valid for each of the Lagrangians $\mathscr{L}^G$ and $\mathscr{L}^M$ themselves.

B. The Material and Geometric Tensors

Define next *matter tensors.*

$$S^{\pi}{}_{\rho\sigma} \stackrel{def}{=} s^{\pi}{}_{\rho\sigma}\big|_{\mathscr{L}=\mathscr{L}^M} \tag{91}$$

(the canonical ST of matter);

$$B^{\gamma\beta\alpha} \stackrel{def}{=} b^{\gamma\beta\alpha}\big|_{\mathscr{L}=\mathscr{L}^M} \tag{92}$$

(the Belinfante tensor for ST **S**);

$$T^{\mu}{}_{\nu} \stackrel{def}{=} t^{\mu}{}_{\nu}\big|_{\mathscr{L}=\mathscr{L}^M} \tag{93}$$

(the canonical EMT of matter)

and analogously for the tensors $\{\overset{add}{S}{}^{\pi}{}_{\rho\sigma}\}$, $\{\overset{mod}{S}{}^{\pi}{}_{\rho\sigma}\}$; $\{\overset{add}{B}{}^{\gamma\beta\alpha}\}$, $\{\overset{mod}{B}{}^{\gamma\beta\alpha}\}$; $\{\overset{add}{T}{}^{\mu}{}_{\nu}\}$, $\{\overset{mod}{T}{}^{\mu}{}_{\nu}\}$, $\{\overset{syn}{T}{}^{\mu\nu}\}$ and $\{\overset{met}{T}{}^{\mu\nu}\}$.

For the above defined matter tensors, relations analogous to those between the total tensors take a place. In particular,

$$\frac{\Delta I^M}{\Delta T^{\alpha}{}_{\beta\gamma}} = \frac{1}{2}\overset{mod}{B}{}^{\gamma\beta}{}_{\alpha}; \tag{94}$$

$$\begin{aligned} 2\frac{\Delta I^M}{\Delta g_{\mu\nu}} &= \overset{met}{T}{}^{\mu\nu} \\ &\equiv \overset{mod}{T}{}^{\mu\nu} - \overset{*}{\nabla}_{\lambda}\overset{mod}{B}{}^{\lambda\mu\nu} + \frac{\Delta I^M}{\Delta\varphi^a}(\Delta^{\mu\nu})^{\,a}{}_{|b}\,\varphi^b. \end{aligned} \tag{95}$$

Now, define the *Cartan tensor* $\mathscr{C} \stackrel{def}{=} \{\mathscr{C}^{[\gamma\beta]}{}_{\alpha} = \mathscr{C}^{\gamma\beta}{}_{\alpha}\}$, the *(generalized)*

Einstein tensor $\mathcal{E} \stackrel{def}{=} \{\mathcal{E}^{(\mu\nu)} \neq \mathcal{E}^{\mu\nu}\}$ and its symmetric part:

$$\frac{-1}{2k}\mathcal{C}^{\gamma\beta}{}_{\alpha} \stackrel{def}{=} \frac{\Delta I^G}{\Delta T^{\alpha}{}_{\beta\gamma}} = \frac{1}{2}\, b^{\gamma\beta}{}_{\alpha}\big|_{\mathscr{L}=\mathscr{L}^G} \tag{96}$$

$$\begin{aligned}\frac{-1}{2k}\mathcal{E}^{\mu}{}_{\nu} &\stackrel{def}{=} \frac{1}{2}\, t^{\mu}{}_{\nu}\big|_{\mathscr{L}=\mathscr{L}^G} \\ &= \frac{1}{2}\left(\mathscr{L}^G\delta^{\mu}_{\nu} - {}^{(G)}G^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma\nu}\right)\end{aligned} \tag{97}$$

$$\frac{-1}{2k}\left(\mathcal{E}^{(\mu\nu)} - \overset{*}{\nabla}_{\lambda}\mathcal{C}^{\lambda(\mu\nu)}\right) \stackrel{def}{=} \frac{\Delta I^G}{\Delta g_{\mu\nu}} = \frac{1}{2}\,\overset{met}{t}{}^{\mu\nu}\Big|_{\mathscr{L}=\mathscr{L}^G} \tag{98}$$

Here,

$${}^{(G)}G_{\alpha}{}^{\beta\gamma\delta} \stackrel{def}{=} G_{\alpha}{}^{\beta\gamma\delta}\big|_{\mathscr{L}=\mathscr{L}^G} = 2\frac{\partial\mathscr{L}^G}{\partial R^{\alpha}{}_{\beta\gamma\delta}}; \tag{99}$$

$k \stackrel{def}{=} (D-1)\Omega_{(D-1)}\,\varkappa$; $\Omega_{(D-1)}$ is an area of $(D-1)$-dimensional unit sphere, and $\varkappa$ is the Newtonian gravitational constant in $(D+1)$-dimensional space-time.

A restriction of the 2-nd Klein identity (66) and the definition (61) to the case of the Lagrangian $\mathscr{L} = \mathscr{L}^G$ gives, with taking into account the definitions (96) - (97), the

Statement 22 *The antisymmetric part of the generalized Einstein tensor is the divergence of the antisymmetric part of the Cartan tensor:*

$$\mathcal{E}^{[\mu\nu]} \equiv \overset{*}{\nabla}_{\lambda}\mathcal{C}^{\lambda[\mu\nu]}. \tag{100}$$

Using the identity (100), one can represent (98) in the form:

$$2\frac{\Delta I^G}{\Delta g_{\mu\nu}} \equiv \frac{-1}{k}\left(\mathcal{E}^{\mu\nu} - \overset{*}{\nabla}_{\lambda}\mathcal{C}^{\lambda\mu\nu}\right). \tag{101}$$

The formulae (79) and (75) rewritten for the Lagrangian $\mathscr{L}^G$ after using the definitions (97) and (96) lead to the

Statement 23 (The generalized twice contracted Bianchi identity) *The generalized Einstein tensor $\mathcal{E}$ and the Cartan tensor $\mathcal{C}$ obey the identity*

$$\overset{*}{\nabla}_{\mu}\mathcal{E}^{\mu}{}_{\nu} \equiv -\mathcal{E}^{\mu}{}_{\lambda}T^{\lambda}{}_{\mu\nu} - \mathcal{C}^{\pi}{}_{\rho\sigma}R^{\rho\sigma}{}_{\pi\nu}. \tag{102}$$

C. The gravitational field equations in the split form

By the equations (101) and definition of the tensor $\{\overset{met}{T}{}^{\mu\nu}\}$, the equations of motion of the metric field, $\Delta(I^G + I^M)/\Delta g_{\mu\nu} = 0$, can be rewritten as

$$\mathcal{E}^{\mu\nu} - \overset{*}{\nabla}_{\lambda}\mathcal{C}^{\lambda\mu\nu} = k\overset{met}{T}{}^{\mu\nu}. \tag{103}$$

If here one takes into account the equations of motion for the torsion field

$$\frac{\Delta(I^G + I^M)}{\Delta T^{\nu}{}_{\mu\lambda}} = 0 \qquad \Leftrightarrow \qquad \mathcal{C}^{\lambda\mu\nu} = k\overset{mod}{B}{}^{\lambda\mu\nu} \tag{104}$$

and the equations of motion for the φ-fields: $\Delta I^M/\Delta\varphi^a = 0$, one can transform (103) to the equation for the metric field only:

$$\mathcal{E}^{\mu\nu} = k\overset{mod}{T}{}^{\mu\nu} \qquad \text{(on the } \phi\text{-equations)}. \tag{105}$$

Now, turn to (104). After antisymmetrization in indexes μ and ν, using the formula (55), the definitions of the tensors $\{\overset{mod}{B}{}^{\gamma\beta\alpha}\}$, $\{\overset{mod}{S}{}^{\gamma,\beta\alpha}\}$, and the identity

$$\Delta^{\overline{\gamma[\beta\alpha]}}_{\lambda\mu\nu} = -\frac{1}{4}\delta^{\gamma}_{\lambda}\delta^{\beta\alpha}_{\mu\nu}, \tag{106}$$

the equation (104) acquires an equivalent form:

$$-2\mathcal{C}^{\lambda[\mu\nu]} = k\overset{mod}{S}{}^{\lambda,\mu\nu}. \tag{107}$$

Thus, the next is valid:

Statement 24 *In an arbitrary metric-torsion field theories the total system of the field equations*

$$\begin{cases} \Delta I/\Delta g_{\mu\nu} = 0; & (108) \\ \Delta I/\Delta T^{\lambda}{}_{\mu\nu} = 0; & (109) \\ \Delta I/\Delta \varphi^{a} = 0 & (110) \end{cases}$$

always may be represented in the Einstein-Cartan-Theory-like form

$$\begin{cases} \mathscr{E}^{\mu\nu} = k \overset{mod}{T}{}^{\mu\nu} \quad \textit{(the g-equations)}; & (111) \\ -2\mathscr{C}^{\lambda}{}_{[\mu\nu]} = k \overset{mod}{S}{}^{\lambda}{}_{\mu\nu} \quad \textit{(the T-equations)}; & (112) \\ \Delta I^{M}/\Delta \varphi^{a} = 0 \qquad \textit{(the } \varphi\textit{-equations)}. & (113) \end{cases}$$

The interpretation of the gravitational equations of the system is as follows. The source of the metric field $\mathbf{g}$ *is the modified canonical EMT of matter* $\overset{mod}{\mathbf{T}}$*, whereas the source of the torsion field* $\mathbf{T}$ *is the modified canonical ST of matter* $\overset{mod}{\mathbf{S}}$*.*

V. BOUNDARY TERMS AND KILLING VECTORS

A. Corrections Induced by Boundary Terms

In the Sec. II D it has been shown that the symmetrized EMT $\overset{sym}{\mathbf{t}}$, consequently the *symmetrized* current $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\xi}]$ also, do not depend on divergences in Lagrangian. Now, let us analyze the problem how a total divergence in the Lagrangian influences the *canonical* current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (12) and superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (30).

Let $\{\mathscr{L}^\mu\}$ be an arbitrary vector constructed from field variables and their derivatives, and let

$$\mathscr{L}' = \mathscr{L} + \Delta\mathscr{L}, \quad \Delta\mathscr{L} \stackrel{def}{=} \overset{*}{\nabla}_\mu \mathscr{L}^\mu \tag{114}$$

be a new Lagrangian with correspondent new current and superpotential:

$$\mathbf{J}'[\delta\boldsymbol{\xi}] = \mathbf{J}[\delta\boldsymbol{\xi}] + \Delta\mathbf{J}[\delta\boldsymbol{\xi}], \tag{115}$$

$$\boldsymbol{\theta}'[\delta\boldsymbol{\xi}] = \boldsymbol{\theta}[\delta\boldsymbol{\xi}] + \Delta\boldsymbol{\theta}[\delta\boldsymbol{\xi}] \tag{116}$$

Of course, both new $\mathbf{J}'[\delta\boldsymbol{\xi}]$ and old $\mathbf{J}[\delta\boldsymbol{\xi}]$ satisfy the Klein-Noether boundary theorem. Therefore from the formulae (31) and (28) rewritten for new quantities it follows

Statement 25 *A total divergence in the Lagrangian induces the changes of the canonical current* $\Delta\mathbf{J}[\delta\boldsymbol{\xi}]$ *and superpotential* $\Delta\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$, *which are connected as*

$$\Delta J^\mu[\delta\boldsymbol{\xi}] = \overset{*}{\nabla}_\nu \left(\Delta\theta^{\mu\nu}[\delta\boldsymbol{\xi}]\right) + \frac{1}{2}\left(\Delta\theta^{\rho\sigma}[\delta\boldsymbol{\xi}]\right) T^\mu{}_{\rho\sigma}. \tag{117}$$

Define an explicit expression for $\Delta\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$. Repeating the logic presented in Secs. II A–II C for the Lagrangian of the type $\mathscr{L} = \overset{*}{\nabla}_\mu \mathscr{L}^\mu$, one obtains

Statement 26 *The addition of the item* $\Delta\mathscr{L} = \overset{*}{\nabla}_\mu \mathscr{L}^\mu$ *to the Lagrangian induces adding the term* $\Delta\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ *to the canonical superpotential, which does not depend on the structure of the vector* $\{\mathscr{L}^\mu\}$ *and is defined in a unique way:*

$$\Delta\theta^{\mu\nu}[\delta\boldsymbol{\xi}] = -2\mathscr{L}^{[\mu}\delta\xi^{\nu]}. \tag{118}$$

This generalizes the statement in [17] to the case of the Riemann-Cartan space.

B. The currents on the Killing vectors

Generalized charges $Q[\chi]$, currents $\mathbf{J}[\chi]$, $\overset{sym}{\mathbf{J}}[\chi]$ and superpotentials $\theta[\chi]$, constructed with using the Killing vectors χ of the Riemann-Cartan space are extremely important in applications.

A definition of the Killing vector field $\chi \overset{def}{=} \{\chi^{\mu}\}$ in Riemann-Cartan space, of course, is more complicated, than in the Riemannian geometry, although is defined by the Lie transport also. Thus,

Definition 1 *The vector field χ is called as the Killing one in the Riemann-Cartan space $\mathscr{C}(1, D)$, if under the Lie translation along the vector χ both the metric tensor* $\mathbf{g}$ *and the torsion tensor* $\mathbf{T}$ *are invariant, that is*

$$\begin{cases} \mathcal{L}_{\chi} g_{\mu\nu} = 0; & (119) \\ \mathcal{L}_{\chi} T^{\lambda}{}_{\mu\nu} = 0. & (120) \end{cases}$$

Let us define an explicit expression for $\overset{sym}{\mathbf{J}}[\chi]$, its physical sense and condition for its conservation. Under the decomposition of the Lagrangian (88) - (90) the current $\overset{sym}{\mathbf{J}}[\chi]$ is decomposed also as

$$\overset{sym}{\mathbf{J}}[\chi] = {}^{(G)}\overset{sym}{\mathbf{J}}[\chi] + {}^{(M)}\overset{sym}{\mathbf{J}}[\chi]; \quad (121)$$

$${}^{(G)}\overset{sym}{\mathbf{J}}[\chi] = -\frac{1}{k}\left[\mathcal{E}^{\mu}{}_{\alpha} + \overset{*}{\nabla}_{\nu}\mathscr{C}^{\mu\nu}{}_{\alpha} + \frac{1}{2}\mathscr{C}^{\rho\sigma}{}_{\alpha}T^{\mu}{}_{\rho\sigma} + \mathscr{C}^{\mu\gamma}{}_{\beta}T^{\beta}{}_{\gamma\alpha}\right]\chi^{\alpha}; \quad (122)$$

$${}^{(M)}\overset{sym}{\mathbf{J}}[\chi] = \overset{sym}{T}{}^{\mu}{}_{\alpha}\chi^{\alpha}. \quad (123)$$

Taking into account the identity (66), one obtains

$${}^{(M)}\overset{sym}{\mathbf{J}}[\chi] = \overset{met}{T}{}^{\mu}{}_{\alpha}\chi^{\alpha} \text{ (on the } \phi\text{-equations)}. \quad (124)$$

This current has a clear physical sense:

Statement 27 *The charges related to the Killing vectors* $\boldsymbol{\chi}_{(a)}$*;* $a = 1, 2, \ldots$

$$Q_{(a)} \stackrel{def}{=} Q[\boldsymbol{\chi}_{(a)}] = \int\limits_{\Sigma} d\sigma_{\mu}\, {}^{(M)}\overset{sym}{J}{}^{\mu}[\boldsymbol{\chi}_{(a)}] \tag{125}$$

define corresponding *conserved quantities in the domain* Σ*, those could be energy, momentum,* etc.

Next, as consequence of the identity (66). one has

$$\overset{sym}{t}{}^{[\mu\nu]} \equiv -\frac{\Delta I}{\Delta \phi^a}(\Delta^{[\mu\nu]})^{\ a}{}_{|b}\,\phi^b. \tag{126}$$

In the case of the Killing vector $\boldsymbol{\chi}$ as a displacement vector, using the identities (83), (126), one easily obtains

$$\overset{*}{\nabla}_{\mu}\ \overset{sym}{J}{}^{\mu}[\boldsymbol{\chi}] = \frac{\Delta I}{\Delta \varphi^a}\mathcal{L}_{\boldsymbol{\chi}}\varphi^a. \tag{127}$$

Decomposing this equality onto pure gravitational and pure matter parts, one proves the next statements.

Statement 28 *The pure gravitational part* ${}^{(G)}\overset{sym}{\mathbf{J}}\ [\boldsymbol{\chi}]$ *of the generalized symmetrized current* $\overset{sym}{\mathbf{J}}\ [\boldsymbol{\chi}]$ *is conserved identically, independently on holding equations of motion at all:*

$$\overset{*}{\nabla}_{\mu}{}^{(G)}\overset{sym}{J}{}^{\mu}[\boldsymbol{\chi}] \equiv 0 \qquad \textit{(off-shell).} \tag{128}$$

Statement 29 *The pure matter part* ${}^{(M)}\overset{sym}{\mathbf{J}}\ [\boldsymbol{\chi}]$ *is conserved identically, if the matter fields* $\boldsymbol{\varphi}$ *are Lie-dragged:*

$$\overset{*}{\nabla}_{\mu}{}^{(M)}\overset{sym}{J}{}^{\mu}[\boldsymbol{\chi}] \equiv 0 \ \textit{(off-shell)} \ \Leftrightarrow \ \mathcal{L}_{\boldsymbol{\chi}}\varphi^a = 0. \tag{129}$$

Statement 30 *The current* ${}^{(M)}\overset{sym}{\mathbf{J}}[\chi]$ *is conserved if only the* φ*-equations hold:*

$$\overset{*}{\nabla}_{\mu}{}^{(M)}\overset{sym}{J}{}^{\mu}[\chi] = 0 \qquad \textit{(on the } \varphi\textit{-equations).} \tag{130}$$

Consequence 2 *Because the current* ${}^{(M)}\overset{sym}{\mathbf{J}}[\chi]$ *is conserved independently on holding both the* g- *and the* T*-equations, then the matter fields can be classified as* external *ones with respect to* g- *and* T*-fields. This gives a possibility to construct and study dynamic variables for the matter fields propagating in a given/fixed Riemann-Cartan space.*

Turn to the generalized canonical current

$$\mathbf{J}[\chi] = {}^{(G)}\mathbf{J}[\chi] + {}^{(M)}\mathbf{J}[\chi]. \tag{131}$$

The pure matter part of this ${}^{(M)}\mathbf{J}[\chi]$ is very cumbersome, see [3]. The expression for the pure gravitational part ${}^{(G)}\mathbf{J}[\chi]$ is, converse, very simple:

$${}^{(G)}\mathbf{J}[\chi] = \mathscr{L}^{G}\chi^{\mu}. \tag{132}$$

Conditions for conservation of these currents one can find in [3].

C. A sense of the generalized superpotential

Consider a pure gravitational part of the canonical charge

$${}^{(G)}Q[\chi] \overset{def}{=} \int\limits_{\Sigma} d\sigma_{\mu}\, {}^{(G)}J^{\mu}[\chi]. \tag{133}$$

Using the Klein-Noether Boundary Theorem, one can write

$$ {}^{(G)}Q[\boldsymbol{\chi}] = \int\limits_{\Sigma} d\sigma_{\mu}\, {}^{(G)}\overset{sym}{J}{}^{\mu}[\boldsymbol{\chi}] + \frac{1}{2!}\oint\limits_{\partial\Sigma} ds_{\mu\nu}\, {}^{(G)}\theta^{\mu\nu}[\boldsymbol{\chi}]. \tag{134} $$

Because on-shell ${}^{(G)}\overset{sym}{\mathbf{J}}[\boldsymbol{\chi}] = -{}^{(M)}\overset{sym}{\mathbf{J}}[\boldsymbol{\chi}]$ one has for the last term here

$$ \begin{aligned} &\frac{1}{2!}\oint\limits_{\partial\Sigma} ds_{\mu\nu}\, {}^{(G)}\theta^{\mu\nu}[\boldsymbol{\chi}] \qquad \text{(on-shell)} \\ &\quad = \int\limits_{\Sigma} d\sigma_{\mu}\, \left({}^{(G)}J^{\mu}[\boldsymbol{\chi}] + {}^{(M)}\overset{sym}{J}{}^{\mu}[\boldsymbol{\chi}] \right). \end{aligned} \tag{135} $$

Substituting into the right hand side the expressions ${}^{(G)}\mathbf{J}[\boldsymbol{\chi}]$ (132) and ${}^{(M)}\overset{sym}{\mathbf{J}}[\boldsymbol{\chi}]$ (124), one obtains

$$ \frac{1}{2!}\oint\limits_{\partial\Sigma} ds_{\mu\nu}\, {}^{(G)}\theta^{\mu\nu}[\boldsymbol{\chi}] = \int\limits_{\Sigma} d\sigma_{\mu} \left[\mathscr{L}^{G}\chi^{\mu} + \overset{met}{T}{}^{\mu}{}_{\nu}\chi^{\nu} \right] \qquad \text{(on-shell)} \tag{136} $$

This shows what the *matter* source structure concretely is presented by the surface integral of the *gravitational* superpotential ${}^{(G)}\boldsymbol{\theta}[\boldsymbol{\chi}]$.

The gravitational theories with Lagrangians of the type

$$ \mathscr{L}^{G} = \mathscr{L}_{(n)}(\mathbf{g}, \mathbf{R}) - \Lambda \tag{137} $$

are very important now. Here, Λ means the cosmological constant, whereas $\mathscr{L}_{(n)}$ is homogeneous in the curvature tensor $\mathbf{R}$ function of the degree n. For example, the Einstein-Cartan theory, curvature squared theories, the pure Lovelock-Cartan theories are of this type. It s easily to show

Statement 31 *In the theories with the Lagrangians of the type* (137), *the generalized charge* ${}^{(G)}Q[\boldsymbol{\chi}]$ *is expressed only through the cosmological*

constant Λ *and the EMT of the matter fields* $\overset{met}{\mathbf{T}}$*:*

$$\begin{aligned} {}^{(G)}Q[\boldsymbol{\chi}] &= \frac{1}{2!}\oint\limits_{\partial\Sigma} dS_{\mu\nu}\,{}^{(G)}\theta^{\mu\nu}[\boldsymbol{\chi}] \\ &= \int\limits_{\Sigma} d\sigma_{\mu}\,\tfrac{2n}{(D+1)-2n}\Lambda\chi^{\mu} \\ &+ \int\limits_{\Sigma} d\sigma_{\mu}\left[\overset{met}{T}{}^{\mu}{}_{\nu} + \tfrac{1}{(D+1)2n}\,\overset{met}{T}\,\delta^{\mu}_{\nu}\right]\chi^{\nu} \\ &\overset{def}{=} Q_{Vac}[\boldsymbol{\chi}] + Q_{Sour}[\boldsymbol{\chi}]. \end{aligned} \tag{138}$$

The term, $Q_{Vac}[\boldsymbol{\chi}]$, describes dynamic characteristics (energy, momentum, ...) of the gravitational vacuum polarized by the matter sources (ϕ-fields). The second term, $Q_{Sour}[\boldsymbol{\chi}]$, describes dynamic characteristics of disturbances, induced by the matter sources, under the vacuum. One has

Statement 32 *Every of the charges* $Q_{Vac}[\boldsymbol{\chi}]$ *and* $Q_{Sour}[\boldsymbol{\chi}]$ *is conserved independently one from another, thus, there is no exchange of dynamic characteristics between the gravitational vacuum and the matter.*

The formula (138) permits to conclude the next.

Consequence 3 *If the cosmological constant is equal to zero,* $\Lambda = 0$*, then* ${}^{(G)}Q[\boldsymbol{\chi}] \neq 0$ *only if* $\overset{met}{\mathbf{T}} \neq 0$*. In such a theory an arbitrary dynamic characteristic of the gravitational vacuum (free gravitational* $\mathbf{g}$*- and* $\mathbf{T}$*-fields without matter) automatically is equal to zero.*

Consequence 4 *The matter influences the gravitational fields by the way that a result of this action brings the information related to the matter dynamic characteristics, which can be identified.*

Consequence 5 *To define dynamic characteristics of matter sources, which fill the domain Σ, it is enough to describe the gravitational fields only at the boundary S of this domain.*

Conclusions

Let us remark that a number of applications of the described above formalism to the popular metric-torsion theories of gravitation are given in Ref. [3]. On the other hand, this paper is based on invited talks given at the Sixth Petrov International Symposium on High Energy Physics, Cosmology and Gravity, 5–20 September, 2013, BITP TIMPANI, Kyiv - Kosivska Poliana, Ukraine.

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ALGEBRAS GROUPS AND GEOMETRIES 31 101 - 116 (2014)

CATEGORIES OF THE BASIC SANTILLI'S ISOSTRUCTURES*

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Received March 2, 2014

Abstract

The objects of modern iso–Geometry, iso-spaces, like all objects of modern iso-Mathematics, are sets of elements of arbitrary nature endowed with some mathematical iso-structure, for example, iso-symmetry etc.

The typical way to think about iso-symmetry is with the concept of a "iso-group". But to get a concept of iso-symmetry that's really up to the demands put on it by modern iso-Mathematics, we need — at the very least — to work with a "category" of iso-symmetries, rather than a iso-group of iso-symmetries. In this article we construct Santilli isofunctor for mathematical iso-structures. The compositions of morphisms between Santilli isofunctors are built in terms of diagrams and equations. There are obtained categories of iso-groups, iso-rings, iso-fields, vector iso-spaces, iso-algebras, etc.

*This research has been partially supported by the R. M. Santilli Foundation.

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1 Santilli iso-functor

The most important mathematical iso-structures are geometric, algebraic and topological ones [1–5].

Santilli's central idea [6, 7] is the generalization of the fundamental unit of the number theory, from its trivial n-dimensional form $I = \text{diag}(1, 1, \ldots)$ to an n-dimensional matrix $\hat{I}$ with the general dependence of all essential variables

$$I = \text{diag}(1, 1, \ldots) \Rightarrow \hat{I} = \hat{I}(s, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \ldots) \tag{1}$$

under the condition of preserving the original axioms of the unit (nondegeneracy, hermiticity, and positive-definiteness).

The "lifting" $I \Rightarrow \hat{I}$ requires, naturally, for necessary compatibility, a generalization of the conventional associative multiplication $x \circ y$ into the so-called isomultiplication

$$x \circ y \Rightarrow x \,\hat{\circ}\, y := xTy\,, \quad T = \text{fixed}\,, \tag{2}$$

where the quantity T is called the *isotopic* element. Then $\hat{I} = T^{-1}$ is a correct left and right unit element of the theory with respect the new multiplication $\hat{\circ}$ and it is called the *isounit.*

DEFINITION 1.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories [8]. A covariant* **Santilli iso-functor** *from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is a family of isofunctions* $\hat{I}$ *[9] which associates to each object* A *in* $\mathbf{X}$ *an object* $\hat{I}A$ *in* $\mathbf{Y}$ *and to each morphism* $f \in \text{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\hat{I}f \in \text{Hom}_{\mathbf{Y}}(\hat{I}A, \hat{I}B)$*, and which is such that:*
(FI) $\hat{I}(g \circ f) = \hat{I}g \circ \hat{I}f$ *for all* $f \in \text{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \text{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\hat{I}\,\text{id}_A = \text{id}_{\hat{I}A}$ *for all* $A \in \text{Ob}\,(\mathbf{X})$.

It is clear from the above that a covariant functor is a transformation that preserves both:

- The domains and the codomains identities.

- The composition of arrows, in particular it preserves the direction of the arrows.

DEFINITION 1.2. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories. A contravariant* **Santilli iso-functor** *from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is a family of isofunctions* $\hat{F}$ *which associates to each object* A *in* $\mathbf{X}$ *an object* $\hat{F}A$ *in* $\mathbf{Y}$ *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\hat{F}f \in \mathrm{Hom}_{\mathbf{Y}}(\hat{F}A, \hat{F}B)$*, and which is such that:*

(FI) $\hat{F}(g \circ f) = \hat{F}f \circ \hat{F}g$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\hat{F}\,\mathrm{id}_A = \mathrm{id}_{\hat{F}A}$ *for all* $A \in \mathrm{Ob}(\mathbf{X})$.

Thus, a contravariant Santilli functor in mapping arrows from one category to the next reverses the directions of the arrows, by mapping domains to codomains and vice versa. A contravariant Santilli functor is also called a Santilli presheaf. These types of Santilli functors will be the principal objects which we will study when discussing Santilli quantum isotheory in the language of topos theory.

DEFINITION 1.3. *Given two categories* C *and* D*, the collection of all covariant (or contravariant) Santilli functors* $F : C \to D$ *is actually a category which will be denoted as* D^C*. This is called the category of Santilli functors and has as objects covariant (or contravariant) Santilli functors and as map natural transformations between Santilli functors.*

2 Categories of Groups and iso-Groups

One of the simplest algebraic structures is the structure of a group [7,8]. A set G of elements of any kind is said to be a *group* if a group operation $a \circ b$ is defined in it satisfying the following axioms:

$G.1^\circ$ For any two elements a and b there exists an element

$$c = a \circ b \tag{3}$$

$G.2°$ This operation is associative, that is, for any three elements a, b, c,

$$(a \circ b) \circ c = a \circ (b \circ c). \tag{4}$$

$G.3°$ There exists a *neutral element* e, i. e. an element such that for every element a,

$$a \circ e = e \circ a = a. \tag{5}$$

$G.4°$ For each element a there exists a *symmetric element* $\bar{a}$ such that

$$a \circ \bar{a} = \bar{a} \circ a = e. \tag{6}$$

If the group operation $a \circ b$ is called *addition*, we write $c = a + b$ and the element c is called the *sum*, the neutral element is called *zero* and is written as 0, the symmetric element is called the *opposite* and is written as $-a$, and the group is called *additive*.

If the group operation $a \circ b$ is called *multiplication*, we write $c = a \cdot b$, or $c = ab$, the element c is called the *product*, the neutral element is called *unit* and is written as 1, the symmetric element is called the *inverse* and is written as a^{-1}, and the group is called *multiplicative*.

If the group satisfies in addition the axiom

$G.5°$. For any two elements a and b

$$a \circ b = b \circ a, \tag{7}$$

then the group is called *commutative* or *Abelian*.

A set of elements endowed with an operation $a \circ b$ without the properties $G.2°$, $G.3°$ and $G.4°$ is called a *magma*. A magma with the property $G.3°$ is called a *unital magma*, a magma with the property $G.2°$ is called a *semigroup*. A magma in which the equations $a \circ x = b$ and $x \circ a = b$ are solvable for all a and b is called a *quasigroup*. A unital semigroup is called a *monoid*, a unital quasigroup is called a *loop*. All these structures (as also the ones to be introduced yet) are termed infinite, respectively finite, if the underlying set is infinite, respectively finite.

DEFINITION 2.1. *A map $f : G \longrightarrow G'$ between two groups $(G, \circ)$ and $(G', \square)$ is called* **homomorphism**, *if the following property holds:*

$$(\forall a, b \in G)[f(a \circ b) = f(a) \square f(b)] \tag{8}$$

Thus a homomorphism "carries" the composition law $\circ$ on G to the composition law $\square$ on G'. Homomorphisms of groups are well visualized in some important aspects with the help of two concepts, the image Im(f) and the kernel Ker(f) of the homeomorphism.

DEFINITION 2.2. *If $f : G \longrightarrow G'$ is a group homomorphism, then we define:*

$$a)\text{Im(f)} = f(a)/a \in G \tag{9}$$

$$b)\text{Ker(f)} = a \in G/f(a) = e' \in G'. \tag{10}$$

It is well known that Im(f) is a subgroup of G' and Ker(f) is a subgroup of G.

DEFINITION 2.3. *A homomorphism f between two groups G and G' is called* **isomorphism** *if f is bijective. In the case where $G = G'$ an homomorphism f is called* **endomorphism** *and an isomorphism is called* **automorphism**.

DEFINITION 2.4. **Grp** *is the category with groups as objects and homomorphisms as morphisms.*

DEFINITION 2.5. *Let* **Grp** *and* **IsoGrp** *be two categories.*
A Santilli iso-functor $\hat{I}$ from **Grp** *associates to each object A in* **Grp** *category an object $\hat{I}A$ in* **IsoGrp** *and to each morphism $f \in \text{Hom}_{\mathbf{Grp}}(A, B)$ a morphism $\hat{I}f \in \text{Hom}_{\mathbf{IsoGrp}}(\hat{I}A, \hat{I}B)$, i.e., we reconstruct the elements of the* **category IsoGrp** *as*

$$\mathbf{Grp} \ni A \longrightarrow \hat{A} \equiv A\hat{I} \in \mathbf{IsoGrp}, \tag{11}$$

where the isounit $\hat{I}$ is defined with the help of an invertible element [6]

$$T : \hat{I} = T^{-1}, \tag{12}$$

called isotopic element, and the new composition law is defined by

$$(\forall \hat{A}, \hat{B} \in \mathbf{IsoGrp})[\hat{A} \hat{\circ} \hat{B} \equiv \hat{A} T \hat{B}]. \tag{13}$$

It can be proved easily, that **IsoGrp** for the fixed isotopic element T, with the above internal composition, can become a iso-group $\hat{G}$ with unit $\hat{I}$ [6]. The notions of isomorphism, iso-isomorphism, etc, can be defined in a similar way as above.

DEFINITION 2.6. *Let* $\hat{\mathcal{I}} : \mathbf{Grp} \to \mathbf{IsoGrp}$ *and* $\hat{\mathcal{I}}' : \mathbf{Grp} \to \mathbf{IsoGrp}$ *be two functors. A* **natural transformation** $\alpha : \hat{\mathcal{I}} \to \hat{\mathcal{I}}'$ *is given by the following data:*

For every object A *in* **Grp** *there is a morphism* $\alpha_A : \hat{\mathcal{I}}(A) \to \hat{\mathcal{I}}'(A)$ *in* **IsoGrp** *such that for every morphism* $f : A \to B$ *in* **Grp** *the following diagram is commutative*

$$\begin{array}{ccc} \hat{\mathcal{I}}(A) & \xrightarrow{\alpha_A} & \hat{\mathcal{I}}'(A) \\ \hat{\mathcal{I}}(f) \downarrow & & \downarrow \hat{\mathcal{I}}'(f) \\ \hat{\mathcal{I}}(B) & \xrightarrow{\alpha_B} & \hat{\mathcal{I}}'(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $\hat{\mathcal{I}}(f) \circ \alpha_A = \alpha_B \circ \hat{\mathcal{I}}'(f)$.

The morphisms α_A, $A \in \mathrm{Ob}(\mathbf{Grp})$, are called the *components of the natural transformation* α.

3 Categories of iso-rings and iso-fields

3.1 IsoRng category

Groups are algebraic systems with one internal composition law. More complicated (and hence, richer) systems are obtained if we introduce a second internal composition law, which is related to the first.

If, in a set of elements of any kind two operations $a+b$ and ab are defined such that

$R.l^\circ$ The set is a commutative group with respect to the operation $a+b$;

$A.2^\circ$ The set is a semigroup with respect to the operation ab;

$R.3^{\circ}$ The operation ab is distributive with respect to the operation $a+b$:

$$a(b+c) = ab + ac, \qquad (a+b)c = ac + bc, \tag{14}$$

the set is called a *ring.*

The ring $\mathbb{Z}$ of integers is a ring with 1, the rings $2\mathbb{Z}$ of even integers, $3\mathbb{Z}$ of integers of the form $3n$, n from $\mathbb{Z}$, and so on, are rings without 1.

DEFINITION 3.1. *A map $f : R \longrightarrow R'$ between two rings $R(+, \star)$ and $R'(\square, \circ)$ is called* **ring homomorphism**, *if it preserves both composition laws, i.e, if the following property holds:*

$$(\forall a, b \in R)[f(a+b) = f(a) \square f(b) \wedge f(a \star b) = f(a) \circ f(b)] \tag{15}$$

DEFINITION 3.2. *A homomorphism f between two rings R and R' is called* **isomorphism** *if f is bijective. In the case where $R = R'$, a homomorphism f is called* **endomorphism** *and an isomorphism is called* **automorphism**.

DEFINITION 3.3. *If $f : R \longrightarrow R'$ is a ring homomorphism, then we define*

$$i)\ Im(f) = f(a) \ / \ a \in R, \tag{16}$$

$$ii)\ Ker(f) = a \in R \ / \ f(a) = e' \in R' \tag{17}$$

where e' is the identity element of the ring R'.

DEFINITION 3.4. **Rng** *is the category with rings as objects and ring homomorphisms as morphisms.*

DEFINITION 3.5. *Let* **Rng** *and* **IsoRng** *be two categories. A Santilli isofunctor $\hat{\mathcal{I}}$ from* **Rng** *associates to each object A in* **Rng** *category an object $\hat{I}A$ in* **IsoRng** *and to each morphism $f \in \mathrm{Hom}_{\mathbf{Rng}}(A, B)$ a morphism $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoRng}}(\hat{I}A, \hat{I}B)$, i.e., we reconstruct the elements of the* **category IsoRng** *as*

$$\mathbf{Rng} \ni A \longrightarrow \hat{A} \equiv A\hat{I} \in \mathbf{IsoRng}, \tag{18}$$

where the isounit $\hat{I}$ is defined with the help of an isotopic element [6]

$$T : \hat{I} = T^{-1}, \tag{19}$$

the first composition law (which is often called addition) is kept unchanged and the second composition law (which is often called multiplication) is defined by

$$(\forall \hat{A}, \hat{B} \in \mathbf{IsoRng})[\hat{A} \hat{\star} \hat{B} \equiv \hat{A} T \hat{B}]. \tag{20}$$

It can be proved easily, that **IsoRng** for the fixed isotopic element T, with the above internal composition, can become a iso-ring $\hat{R}$ with isounit $\hat{I}$ [6]. The notions of isomorphism, iso-isomorphism, etc, between two iso-rings can be defined in a similar way as above.

3.2 IsoField category

A ring in which the set of elements without 0 is a commutative group with respect to the operation *ab* is called a *field.*

DEFINITION 3.6. *A* **field** $\mathbb{F}$ *is a commutative ring with unit where every element (except zero), is invertible. More precisely a field $\mathbb{F}(+, \cdot)$ is:*
1) An abelian group with respect to an internal operation, which is usally denoted with $+$ and called addition and
2) is equipped with a second internal operation, denoted with $\cdot$ and called multiplication, so that the following rules hold

$$i)\ (\forall\ \alpha, \beta, \gamma \in \mathbb{F})\ \ [\alpha(\beta\gamma) = (\alpha\beta)\gamma], \tag{21}$$

$$ii)\ (\exists\ 1 \in \mathbb{F})\ (\forall\ \alpha \in \mathbb{F})\ \ [\alpha 1 = 1\alpha = \alpha], \tag{22}$$

$$iii)\ (\forall\ \alpha \in \mathbb{F})\ \ (\exists\ \alpha^{-1} \in \mathbb{F})\ \ [\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1], \tag{23}$$

$$iv)\ (\forall\ \alpha, \beta \in \mathbb{F})\ \ [\alpha\beta = \beta\alpha], \tag{24}$$

$$v)\ (\forall\ \alpha, \beta, \gamma \in \mathbb{F})\ \ [\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma]. \tag{25}$$

A ring in which the set of elements without 0 is a noncommutative group with respect to the operation *ab* is called a *skew field.*

DEFINITION 3.7. *A map* $f : \mathbb{F} \longrightarrow \mathbb{F}'$ *between two fields* $\mathbb{F}(+,\cdot)$ *and* $\mathbb{F}'(\square,\circ)$ *is called* **field homomorphism**, *if it preserves both composition laws, i.e, if the following property holds:*

$$(\forall a, b \in \mathbb{F})[f(a+b) = f(a)\square f(b) \wedge f(a\cdot b) = f(a)\circ f(b)] \qquad (26)$$

In the following we shall denote a field for brevity with $\mathbb{F}$ instead of $\mathbb{F}(+,\cdot)$.

The ring $\mathbb{Q}$ of rational numbers, the ring $\mathbb{R}$ of real numbers, the ring $\mathbb{C}$ of complex numbers, the ring $\mathbb{F}_p$ of residues modulo a prime integer p, and the *Galois field* $\mathbb{F}_q$ $(q = p^k)$ obtained from $\mathbb{F}_p$ by adjoining the roots of an irreducible algebraic equation of degree k over $\mathbb{F}_p$, are fields. The ring $\mathbb{H}$ of *quaternions* is a skew field. The fields $\mathbb{F}_p$ and $\mathbb{F}_q$ are finite and contain p and q= p^k elements, respectively. The rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$ are infinite, and the three last fields are *continuous* (*topological*). A field $\mathbb{Q}(\alpha, \beta, \ldots)$ of algebraic numbers, obtained from $\mathbb{Q}$ by adjoining the roots $\alpha, \beta, \ldots$ of an irreducible algebraic equation over $\mathbb{Q}$, and the field $\mathbb{Q}_p$, of *p-adic numbers*, which is also an extension of $\mathbb{Q}$, are also infinite, and the last field is continuous.

DEFINITION 3.8. **Field** *is the category with fields as objects and field homomorphisms as morphisms.*

DEFINITION 3.9. *Let* **Field** *and* **IsoField** *be two categories. A Santilli isofunctor* $\hat{\mathcal{I}}$ *from* **Field** *associates to each object* A *in* **Field** *category an object* $\hat{I}A$ *in* **IsoField** *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{Field}}(A, B)$ *a morphism* $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoField}}(\hat{I}A, \hat{I}B)$, *i.e., we reconstruct the elements of the* **category IsoField** *as*

$$\mathbf{Field} \ni A \longrightarrow \hat{A} \equiv A\hat{I} \in \mathbf{IsoField}, \qquad (27)$$

where the isounit $\hat{I}$ *is defined with the help of an isotopic element [6]*

$$T : \hat{I} = T^{-1}, \qquad (28)$$

the first composition law (which is often called addition) is kept unchanged and the second composition law (which is often called multiplication) is defined by

$$(\forall\, \hat{A}, \hat{B} \in \mathbf{IsoField})[\hat{A}\hat{\cdot}\hat{B} \equiv \hat{A}T\hat{B}]. \tag{29}$$

4 Subcategories, subisogroups, subisorings

Subsets of categories, isogroups and isoringsthat are also categories, isogroups and isorings respectively, are called *subcategories*, *subisogroups* and *subisorings*, respectively.

DEFINITION 4.1. *A category* $\mathbf{D}$ *is called a* **subcategory** *of a category* $\mathbf{C}$ *if* $\mathrm{Ob}(\mathbf{D}) \subseteq \mathrm{Ob}(\mathbf{C})$, $\mathrm{Mor}(\mathbf{D}) \subseteq \mathrm{Mor}(\mathbf{C})$, *and morphism composition in* $\mathbf{D}$ *coincide with their composition in* $\mathbf{C}$.

A bijection between two sets endowed with algebraic structures which preserves the operations of these structures is called an *isomorphism*, sets between which there is an isomorphism are called *isomorphic*.

A surjection or injection between two sets endowed with algebraic structures which preserves the operations is called a *homomorphism*, sets between which there is a homomorphism are called *homomorphic*.

If two isogroups $\hat{G}$ and $\hat{H}$ are homomorphic, the elements of $\hat{G}$ corresponding to the neutral element of $\hat{H}$ form a subisogroup $\hat{N}$ of $\hat{G}$, such a subisogroup is called an *invariant subisogroup* or *normal subisogroup*. In this case the isogroup isomorphic to $\hat{H}$ is called the *quotient isogroup* and is written as $\hat{G}/\hat{N}$.

If two isorings $\hat{\mathcal{R}}$ and $\hat{\mathcal{S}}$ are homomorphic, the elements of $\hat{\mathcal{R}}$ corresponding to the zero in $\hat{\mathcal{S}}$ form a subisoring $\hat{\mathcal{J}}$ of $\hat{\mathcal{R}}$ called an *ideal*. In this case the isoring isomorphic to $\hat{\mathcal{S}}$ is called the *quotient isoring* and is written as $\hat{\mathcal{R}}/\hat{\mathcal{J}}$.

Isomorphisms of a isogroup or a isoring onto itself are called *isogroup* or *isoring automorphisms*, respectively; homomorphisms of a isogroup or a isoring into itself are called *isogroup* or *isoring endomorphisms*, respectively.

5 Direct sums and products

If $\hat{G}$ and $\hat{H}$ are two isogroups with operations $\hat{g}_1\hat{\circ}\hat{g}_2$ and $\hat{h}_1\hat{\circ}\hat{h}_2$, the pairs $(\hat{g},\hat{h})$ with operation $(\hat{g}_1,\hat{h}_1)\hat{\circ}(\hat{g}_2,\hat{h}_2) = (\hat{g}_1\hat{\circ}\hat{g}_2,\hat{h}_1\hat{\circ}\hat{h}_2)$ form a isogroup. If $\hat{G}$ and $\hat{H}$ are both additive, respectively multiplicative, isogroups, this isogroup is called a *direct sum*, respectively *direct product*, and is written as $\hat{G}\hat{\oplus}\hat{H}$, respectively $\hat{G}\hat{\otimes}\hat{H}$.

The direct sum $\hat{\mathcal{R}}\hat{\oplus}\hat{\mathcal{S}}$ of rings $\hat{\mathcal{R}}$ and $\hat{\mathcal{S}}$ is similarly defined.

6 Vector isospaces

A set L^n of elements of any kind, called *vectors*, is said to be an *n-dimensional vector space* if in this set the operations of *addition* and of *multiplication by scalars*, that is real numbers, are defined, satisfying:

$VI.1^\circ$ - 5° Addition of vectors satisfies axioms $G.1-5^\circ$ for a commutative group;

$VII.1^\circ$ For any vector a and any scalar λ there exists a vector

$$\mathbf{b} = \mathbf{a}\cdot\lambda = \mathbf{a}\lambda \tag{30}$$

called the product of $\mathbf{a}$ by λ;

$VII.2^\circ$ Multiplication by 1 does not change a vector:

$$\mathbf{a}\cdot 1 = \mathbf{a}; \tag{31}$$

$VII.3^\circ$ Multiplication of vectors by scalars is distributive with respect to addition of scalars:

$$\mathbf{a}(\lambda+\mu) = \mathbf{a}\lambda + \mathbf{a}\mu; \tag{32}$$

$VII.4^\circ$ Multiplication of vectors by scalars is distributive with respect to addition of vectors:

$$(\mathbf{a}+\mathbf{b})\lambda = \mathbf{a}\lambda + \mathbf{b}\lambda; \tag{33}$$

$VII.5^\circ$ Multiplication of vectors by scalars is associative:

$$(\mathbf{a}\lambda)\mu = \mathbf{a}(\lambda\mu); \tag{34}$$

and *axioms* $VIII.1^\circ$ - 2° *of dimension*, which are based on the notions of linear independence and dependence of vectors. Vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are said to be *linearly independent* if a linear combination $\mathbf{a}_1\lambda_1 + \mathbf{a}_2\lambda_2 + \ldots + \mathbf{a}_m\lambda_m$ is equal to zero only if all coefficients $\lambda_i = 0$, and *linearly dependent* if there are nonzero coefficients λ_i such that this linear combination is equal to zero.

$VIII.1^\circ$ There exist n linearly independent vectors;

$VIII.2^\circ$ Any $n+1$ vectors are linearly dependent.

If we have chosen n linearly independent vectors $\mathbf{e}_1\, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in L_n, then each vector can be written as

$$\mathbf{x} = \sum_i \mathbf{e}_i x^i = \mathbf{e}_i x^i. \tag{35}$$

The numbers x^i are called the *coordinates* of the vector $\mathbf{x}$, the vectors $\mathbf{e}_i$ are called *basis vectors*. Later we will write the sums (35) only in the last form and when in our formulas the same upper and lower indices appear we will always mean summation with respect to these indices.

DEFINITION 6.1. *A subset U of a vector space V is called* **vector subspace** *if it is a subsystem which obeys the axioms of vector space in itself, that is U is closed under vector addition and scalar multiplication.*

DEFINITION 6.2. *The notions of* Ker(f) *and* Im(f) *are defined by the relations*

$$a)\ \mathrm{Ker(f)} = [x \in V \ / \ f(x) = 0 \in U], \tag{36}$$

$$b)\ \mathrm{Im(f)} = [f(x) \in U \ / \ x \in V]. \tag{37}$$

It easy proved that Ker(f) and Im(f) are subspaces of V and U respectively.

DEFINITION 6.3. *Let V and U two vector spaces over the same field $\mathbb{F}$ (not necessarily of the same dimension). A map $f : V \longrightarrow U$ is called* **linear map** *or* **linear transformation** *if the following property is holds:*

$$(\forall \alpha, \beta \in \mathbb{F})\,(\forall x, y \in V)\,[f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)]. \tag{38}$$

In case $V = U$, the map f is called **linear operator**.

Two vector spaces are called *isomorphic* if there is a bijection between them that preserves addition of vectors and multiplication of vectors by scalars, and *homomorphic* if there is a surjection or an injection between them that preserves these operations. Isomorphisms of a vector space onto itself and homomorphisms of a vector space into itself are called *automorphisms* and *endomorphisms* of this vector space respectively. Automorphisms and endomorphisms of a vector space are called *linear transformations* in it.

DEFINITION 6.4. **$\mathbf{Vect}_k$ category** *consisting of vector spaces over a field k as objects and k-linear maps as morphisms;*

From the definition of vector space one can see that we cannot construct an isotopy of a vector space without first introducing an isotopy of the field, because the multiplicative unit I of the space is that of the underlying field. Note that we are lifting of the field, but the elements of the linear space remain unchanged.

DEFINITION 6.5. *Let* **Vect** *and* **IsoVect** *be two categories, V be a vector space over the field $\mathbb{F}$ and $\hat{\mathbb{F}}$ an isofield of $\mathbb{F}$. A Santilli iso-functor $\hat{\mathcal{I}}$ from* **Vect** *associates to each object A in* **Vect** *category an object $\hat{I}A$ in* **IsoVect category** *by "isovector space" as the vector space $\hat{V}$, (which is the same set as V), over the isofield $\hat{\mathbb{F}}$ equipped with a new external operation $\diamond$ which is such to verify all the axioms for a vector space, i.e,*

$$(\forall\ \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ x \in \hat{V})[\hat{\alpha} \diamond (\hat{\beta} \diamond x) = (\hat{\alpha} \star \hat{\beta}) \diamond x], \tag{39}$$

$$(\forall\ \hat{\alpha}, \in \hat{\mathbb{F}})\ \ (\forall\ x, y \in \hat{V})[\hat{\alpha} \diamond (x + y) = (\hat{\alpha} \diamond x + \hat{\beta}) \diamond y], \tag{40}$$

$$(\forall\ \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ x \in \hat{V})[(\hat{\alpha} + \hat{\beta}) \diamond x) = (\hat{\alpha} \diamond x + \hat{\beta}) \diamond x], \tag{41}$$

$$(\forall\ x \in V)[\hat{I} \diamond x = x \diamond \hat{I} = x]. \tag{42}$$

and to each morphism $f \in \mathrm{Hom}_{\mathbf{Vect}}(A, B)$ an isolinear transformation as a morphism $\hat{I}f \in \mathrm{Hom}_{\mathbf{IsoVect}}(\hat{I}A, \hat{I}B)$, which is an isomap:

$$\hat{f} : \hat{V} \longrightarrow \hat{V}', \tag{43}$$

between two isolinear vector spaces $\hat{V}$ *and* $\hat{V}'$ *over the same isofield* $\hat{\mathbb{F}}$ *which preserves the sum and isomultiplication, i.e., which is such that*

$$(\forall\ \hat{\alpha},,\hat{\beta} \in \hat{\mathbb{F}})\ \ (\forall\ x, y \in V)[\hat{f}(\hat{\alpha} \star x + \hat{\beta} \star y) = \hat{\alpha} \star \hat{f}(x) + \hat{\beta} \star \hat{f}(y)]. \quad (44)$$

7 Algebras

If the set A is simultaneously a vector space L^n and a ring $\mathcal{R}$ or a field $\mathbb{F}$ and if the following axiom holds:

$A.1^\circ$ For any two elements a and b and any two scalars λ and μ we have

$$(a\lambda)(b\mu) = (ab)(\lambda\mu), \quad (45)$$

then the set A is called an *n-algebra.*

The product of two basis elements of an algebra can be written as

$$e_i e_j = C_{ij}^k e_k. \quad (46)$$

The numbers C_{ij}^k are called the *structure constants* of the algebra. In this book we will consider both associative and nonassociative algebras. The field $\mathbb{C}$ of complex numbers is a commutative associative algebra with basis $1, i$ $(i^2 = -1)$, the skew field $\mathbb{H}$ of quaternions is a noncommutative associative algebra with basis $1, i, j, k$ $(i^2 = j^2 = k^2 = -l,\ ij = -ji = k,$ $jk = -kj = i,\ ki = -ik = j)$.

An algebra which is a field or a skew field is called a *division algebra.*

Two algebras are called *isomorphic* if there is a bijection between them that preserves addition and multiplication of vectors, and multiplication of vectors by scalars, and *homomorphic* if there is a surjection or an injection between them that preserves these operations. Isomorphisms of an algebra onto itself and homomorphisms of an algebra into itself are *algebra automorphisms* and *algebra endomorphisms* of this algebra, respectively.

DEFINITION 7.1. *A subset* V *of an algebra* A *is called a subalgebra if it satisfies the algebra axioms, i.e., if it is an algebra in its own right.*

It is easy to proove that a subset $\mathsf{V} \subset \mathsf{A}$ of an algebra A is a subalgebra iff:

$$(\forall x, y \in \mathsf{A})(\forall \alpha \in \mathbb{F})[x + y \in \mathsf{A}, \alpha x \in \mathsf{A}, xy \in \mathsf{A}] \tag{47}$$

that is, the subset V is closed under all the composition laws.

DEFINITION 7.2. *A map* $f : \mathsf{A} \longrightarrow \mathsf{A}'$ *between algebras over the same field* $\mathbb{F}$ *is called* **morphism** *if:*

$$(\forall x, y \in \mathsf{A})(\forall \alpha, \beta \in \mathbb{F})[f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \wedge f(xy) = f(x)f(y)]. \tag{48}$$

DEFINITION 7.3. *The usual statements about the image and kernel of a morphisms (48) hold.*

DEFINITION 7.4. **Alg** *is the category with algebras as objects and maps (48) as morphisms.*

DEFINITION 7.5. *Let* **Alg** *and* **IsoAlg** *be two categories.*
A Santilli iso-functor $\hat{\mathcal{I}}$ *from* **Alg** *associates to each object* A *in* **Alg** *category an object* $\hat{\mathsf{A}}$ *in* **IsoAlg** *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{Alg}}(\mathsf{A}, \mathsf{B})$ *a morphism* $\hat{f} \in \mathrm{Hom}_{\mathbf{IsoAlg}}(\hat{\mathsf{A}}, \hat{\mathsf{B}})$, *i.e., we reconstruct the elements of the* **category IsoAlg** *as an "isotope"* $\hat{\mathsf{A}}$ *of an algebra* A *with elements* $x, y, z, \cdots$ *and product* xy *over a field* $\mathbb{F}$, *is the same vector space* L^n *but defined over the isofield* $\hat{\mathbb{F}}$, *equipped with a new product* $x \star y$, *called "isotopic product", which is such to verify the original axioms of* A.

A possible way to overcome basic issues in present-day Category Formulation of Santilli's Isotopies is to solve another three problems:

Problem 1, it has to be formulated the basic results of categories of topological, incidence and metric isostructures: Category of topological isospaces; Category of Affine isospaces; Category of projective isospaces; Category of Euclidean isospaces; Category of pseudo-Euclidean isospaces; Category of conformal isospaces; Category of pseudoconformal isospaces.

Problem 2, it has to be shown how Lie-Santilli Isogroups and Isoalgebras. can be represented in category theory.

Problem 3, it has to be described the categorical concept of infinite-dimensional isospaces: Category of infinite-dimensional linear isospaces; Category of Hilbert isospaces; Category of Banach isospaces and Categories of Hilbert and Banach isoalgebras.

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